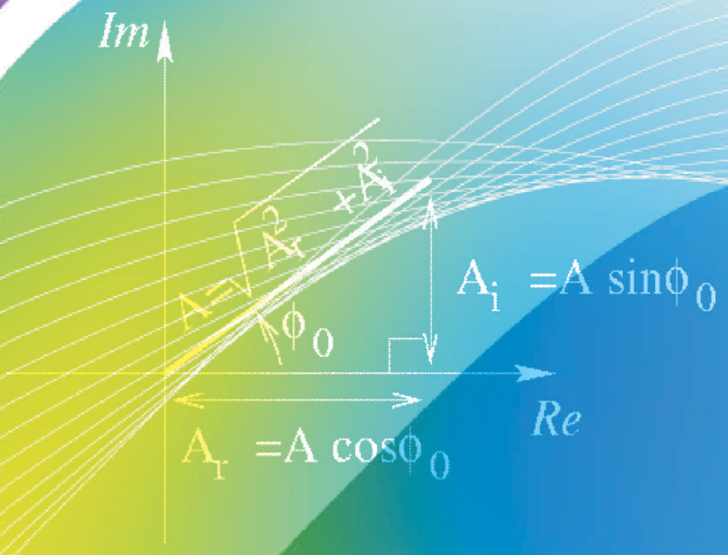
NEW AGE

THIRD EDITION

THE ELEMENTS OF COMPLEX ANALYSIS



B. Choudhary

NEW AGE INTERNATIONAL PUBLISHERS

THE ELEMENTS
OF
COMPLEX ANALYSIS

This page intentionally left blank

THE ELEMENTS
OF
COMPLEX ANALYSIS

(THIRD EDITION)

B. Choudhary
Former
Professor, Department of Mathematics
Indian Institute of Technology, Delhi

Committed to Educate the Nation

Publishing Globally
IN INDIA
NEW AGE INTERNATIONAL (P) LIMITED, PUBLISHERS
LONDON • NEW DELHI • NAIROBI
Bangalore • Chennai • Cochin • Guwahati • Hyderabad • Kolkata • Lucknow • Mumbai
Visit us at **www.newagepublishers.com**

Copyright © 2021, 2013, 1983, New Age International (P) Ltd., Publishers

Published by New Age International (P) Ltd., Publishers

All rights reserved.
No part of this ebook may be reproduced in any form, by photostat, microfilm, xerography, or any other means, or incorporated into any information by retrieval system, electronic or mechanical, without the written permission of the publisher. *All inquiries should be emailed to* ***rights@newagepublishers.com***

This ebook has been given to EBSCO for hosting on non-exclusive basis.

ISBN: 978-81-224-5711-7

PUBLISHING GLOBALLY
NEW AGE INTERNATIONAL (P) LIMITED, PUBLISHERS
7/30A, Daryaganj, New Delhi - 110002
Visit us at **www.newagepublishers.com**

To

My Guides and Teachers:

Late Professor L. S. Bosanquet
Late Professor B. Kuttner
Late Dr. P. Vermes

This page intentionally left blank

PREFACE TO THE THIRD EDITION

This book is meant as a textbook for a course in Complex Analysis at graduate level of universities in India and abroad. It is also intended to be useful for scientists and engineers. The second edition of this book was reviewed by learned mathematicians in many reputed international journals of mathematics and the response was very encouraging. In this third edition we have incorporated reviewer's suggestions and comments. We welcome any further comments and suggestions from readers. Finally, I thank the publisher for his cooperation in bringing out the book.

B. Choudhary

This page
intentionally left
blank

PREFACE TO THE SECOND EDITION

Complex Analysis was originally developed for the sake of its application. Keeping in view the applied nature of the subject I have included in this edition topics like some physical applications of conformal mapping, extension of the maximum modulus principle, some consequences of Jensen's theorem and Hadmard's gap theorem.

The justifiable criticism was that the chapters on conformal mapping and analytic continuation were not given more attention than most other books. In this edition, both these chapters have been strengthened by new results and examples.

Solutions to some more selected exercises which involve lot of new ideas and theoretical considerations have been provided at the end. I hope this will help the students to get actively involved in the subject.

Finally, I thank the publisher for his cooperation in bringing out the book. Readers are urged to send their comments and suggestions so that I can improve the usefulness of the book.

B. Choudhary

This page intentionally left blank

PREFACE TO THE FIRST EDITION

This book is an introductory course in Complex Analysis of one Complex Variable. It contains sufficient material for one semester course.

The prerequisite for this book is a course in advanced calculus and a course in elementary modern algebra.

The book consists of fourteen chapters and two appendices. Chapter 1 is very elementary and deals with the field of complex numbers. Assuming that the reader is familiar with metric spaces, continutity, convergence, compactness, connectedness, etc., we briefly introduce these concepts in Chapter 2. Chapter 3 to Chapter 10, includes the essentials of Complex Analysis, which cannot be left out in one semester course.

Since the proof of the Riemann Mapping Theorem is somewhat more difficult than the study of the specific cases considered in Chapter 9, it has been presented in the Appendix. Chapters 12-14 deal with further analytic aspects of functions in many directions, which may lead to some other branches of mathematics. Chapter 11 initiates the reader in the consideration of functions as points in a metric space and presents Weierstrass Factorization Theorem and its application. The homological version of Cauchy's Theorem is presented in the Appendix. Exercises are given at the end of each chapter. Solutions to some selected problems are special feature of the book.

This book grew out of a regular course given by me at M.Sc. level of Indian Institute of Technology, Delhi. I have been influenced by many books on the same subject, especially by L. Ahlfors, H. Cartan, J.B. Conway and Serge Lang. I have also referred, in various places, to those books which have been of particular assistance to me in preparing lecture notes for this book.

This book covers the syllabus on Complex Analysis at graduate level of the universities in India and undergraduate level of the universities in North America and Europe. I have resisted the temptation to include everything of the subject in this text, consequently there are many facets of the subject which have been omitted.

I have chosen the topics with greatest care and have tried to present them systematically with diagrams and illustrations. In each chapter the level grows gradually and accumulations of lengthy theorems at one place have been avoided.

My thanks are due to many people who have read drafts of the text. I am afraid that despite all my efforts, if some of the mistakes are still survived, I will be grateful to the readers who are kind enough to point them out.

B. Choudhary

CONTENTS

Preface to the Third Edition *vii*

Preface to the Second Edition *ix*

Preface to the First Edition *xi*

1. Sets, Functions and Complex Numbers **1**

1.1 Sets 1

1.2 Functions 3

1.3 Countable Sets 5

1.4 Fields 7

1.5 Complex Numbers 8

1.6 The Complex Plane 10

1.7 The Roots of Complex Numbers 11

1.8 Stereographic Projection 14

1.9 Spherical Representation 14

Exercises 15

2. Metric Spaces **19**

2.1 Definition 19

2.2 Convergence, Completeness 23

2.3 Continuous Functions 25

2.4 Compactness 26

2.5 Connectedness 29

Exercises 33

3. Elementary Properties of Analytic Functions **36**

3.1 Limits and Continuity 36

3.2 Complex Differentiability 37

3.3 The Cauchy-Riemann Equations 40

3.4 Exponential Function 42
3.5 Trigonometric and Hyperbolic Functions 44
3.6 Logarithm 45
3.7 Inverse Trigonometric and Hyperbolic Functions 46
Exercises 47

4. Line Integral and Cauchy's Theorem 50

4.1 Definitions 50
4.2 Riemann-Stieltjes Integral 51
4.3 Line-Integral 52
4.4 Modulus of Continuity 53
4.5 Local Primitive 55
4.6 Cauchy's Theorem (Homotopy Form) 59
4.7 Global Primitives 62
Exercises 63

5. Application of Cauchy's Theorem 66

5.1 The Winding Number 66
5.2 Statement of Cauchy's Theorem 69
5.3 Some Consequences of Cauchy's Theorem 70
5.4 Application of Cauchy's Integral Formula 72
Exercises 76

6. Power Series 79

6.1 Infinite Series in C 79
6.2 Series of Functions and Uniform Convergence 82
6.3 Power Series 84
Exercises 88

7. Laurent Series, Singularities 90

7.1 Power Series Representation of Analytic Functions 90
7.2 Taylor Series 91
7.3 Zeros of Analytic Function 92
7.4 Laurent Series 93
7.5 Isolated Singularities 96
7.6 Limit Points of Zeros and Poles 97
7.7 Meromorphic Functions 98
Exercises 100

8. Residue Theorem and Its Applications 102

8.1 Definition 102
8.2 Applications of the Residue Theorem 104
8.3 The Logarithmic Residue 113
Exercises 116

9. Conformal Mapping 119

9.1 Definition 120
9.2 Linear Fractional Transformation 122
9.3 Definition 125
9.4 Symmetry 126
9.5 The Schwarz-Christoffel Transformation 130
9.6 The Transformations $w = \sin z$ and $w = \cos z$ 131
9.7 Riemann Surfaces 132
Exercises 135

10. Harmonic Functions 137

10.1 Definitions of Harmonic Functions 137
10.2 Harmonic Functions and Analytic Functions 138
10.3 Harmonic Functions on a Disk 140
10.4 Construction of Harmonic Functions on a Disk 143
10.5 Some Physical Applications of Conformal Mapping 147
10.6 Some Other Physical Interpretations 151
Exercises 153

11. Weierstrass Factorization Theorem 157

Part I

11.1 Metric on C(G, Ω) 157
11.2 Spaces of Analytic Functions 160
11.3 Weierstrass Factorization Theorem 161
Exercises 170

Part II

Extension of the Maximum Modulus Principle 173

11.4 Hadamard's Product Representation 173
11.5 The Effect of Zeros, Jensen's Formula 174
11.6 Some Consequences of Jensen's Theorem 176
11.7 Phragmen-Lindelöf Theorem 179
11.8 The Gamma Function 182
Exercises 186

12. Elliptic Functions 187

12.1 Groups 187
12.2 Elliptic Functions 188
12.3 Weierstrass' Elliptic Functions 191
12.4 The Addition Theorems 194
12.5 The Weierstrass' Zeta Function 196
12.6 The Weierstrass' Sigma Function 198
Exercises 200

13. Analytic Continuation, Differential Equations 202

13.1 Analytic Continuation 202
13.2 Continuation along a Path 203
13.3 Continuation by Reflection 204
13.4 Nowhere-Continuable Power Series 209
13.5 Differential Equations 213
13.6 Solutions at Infinity 216
13.7 The Hypergeometric Differential Equation 217
13.8 Some Simple Consequences of the Function $F(a, b, c, z)$ in (33) 218
13.9 Bessel's Differential Equation 220
13.10 Legendre's Differential Equation 221
Exercises 222

14. Approximation by Rational Functions and Polynomials 225

14.1 Uniform Approximation 225
14.2 Locally Analytic Functions 230
Exercises 235

APPENDIX 1: Riemann Mapping Theorem 237
APPENDIX 2: Homological Version of Cauchy's Theorem 243
Solutions to Some Selected Exercises 249
***Bibliography* 293**
***List of Symbols* 294**
***Index* 297**

1

SETS, FUNCTIONS AND COMPLEX NUMBERS

1.1 SETS

Let A be a given set. We write $x \in A$ if x is an element of A. The negation of $x \in A$ is written in the form $x \notin A$.

If each element of E is also an element of F, then we say that E is a subset of F, or that E is contained in F, or that F contains E and write

$$E \subset F \quad \text{or} \quad F \supset E.$$

In particular, $E \subset E$ for every set E.

The set which contains no element is called the **empty set.** We denote the empty set by ϕ. Note that the empty set is a subset of every set.

Two sets E and F are said to be equal if they contain the same elements. In order to show that the sets E and F are equal we must show that $E \subset F$ and $F \subset E$. If $E \subset F$ and $F \subset E$, we write

$$E = F.$$

If $E \subset F$ and $E \neq F$, E is called a **proper subset of** F.

Let P denotes a property for a collection of elements. We use the symbol

$$\{x : P\}$$

to denote the set of all elements x which have the property P.

Let A and B be two sets. The **union** of two sets A and B is defined to be the set of all elements which belong either to A or to B or to both A and B. In symbols,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The **intersection** of two sets A and B is defined to be the set of all elements which belong to both A and B. In symbols,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Let A and B be two sets. If $A \cap B = \phi$, then we say that A and B are **disjoint.**

We now list some of the algebraic properties of the operations on sets that we have just defined. The proofs of these assertions are left to the reader.

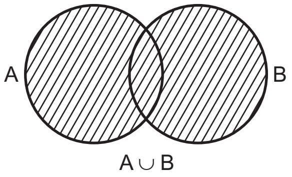


Fig. 1.I

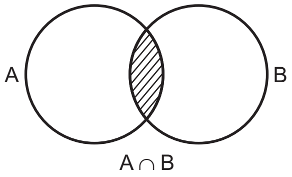


Fig. 1.II

Properties

(*a*) $A \cup A = A, A \cup \phi = A, A \cap A = A, A \cap \phi = \phi$;

(*b*) $A \cup B = B \cup A, A \cap B = B \cap A$;

(*c*) $(A \cup B) \cup C = A \cup (B \cup C), A \cap (B \cap C) = (A \cap B) \cap C$;

(*d*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

These properties are called the **idempotent,** the **commutative,** the **associative,** and the **distributive properties,** respectively, of the operations of union and intersection of sets.

Let A and B be two sets. The complement of B relative to A is the set of all elements of A which are not in B. In symbols,

$$A - B = \{x : x \in A, x \notin B\}.$$

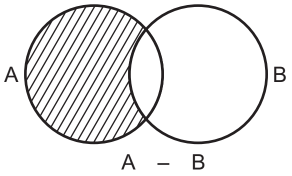


Fig. 1.III

We now state the De Morgan's laws for three sets.

De Morgan's Laws

(*a*) $A - (B \cup C) = (A - B) \cap (A - C)$;

(*b*) $A - (B \cap C) = (A - B) \cup (A - C)$.

We verify the first of the two equalities. If $x \in A - (B \cup C)$, then $x \in A$, $x \notin B$ and $x \notin C$. Hence $x \in A - B$ and $x \in A - C$. This shows that $x \in (A - B) \cap (A - C)$.

Conversely, if $x \in (A - B) \cap (A - C)$, then $x \in (A - B)$ and $x \in (A - C)$. Thus $x \in A$, $x \notin B$ and $x \notin C$. This shows that $x \in A$ and $x \notin B \cup C$, *i.e.* $x \in A - (B \cup C)$.

The verification of the second equalities is left to the reader.

Let A and B be two non-empty sets. The **Cartesian product** $A \times B$ of the sets A and B is defined to be the set of all ordered pairs (x, y) where $x \in A$ and $y \in B$. In symbols,

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}.$$

Example : Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$.

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3), (d, 1), (d, 2), (d, 3)\}.$$

Note that $(a, 1) \in A \times B$, but $(1, a) \notin A \times B$.

We shall use the following notations for designating intervals. If a and b are real numbers such that $a < b$, then we define

$$[a, b] = \{x : a \le x \le b\};$$
$$(a, b) = \{x : a < x < b\};$$
$$[a, b) = \{x : a \le x < b\};$$
$$(a, b] = \{x : a < x \le b\}.$$

1.2 FUNCTIONS

Let A and B be arbitrary non-empty sets. A function from A to B is defined to be a set f of **ordered pairs** in $A \times B$ such that if (x, y) and (x, y') belong to f, then $y = y'$. In other words, a function from A to B is some rule whereby to each element $x \in A$ corresponds a uniquely determined element $y \in B$. The y which corresponds in this way to a given x is denoted by $f(x)$, and is called the **value of** f. The set A is called the **domain of definition** of f and the set of values of f is called the range of f. The terms **"function"** or **"mapping"** are synonymous and we denote them by

$$f : A \to B,$$

with domain A, and range contained in B.

Let A and B be two sets and let $f : A \to B$ be a map. Suppose that D is a subset of A. The image $f(D)$ is the subset of B defined by

$$f(D) = \{f(x) : x \in D\}.$$

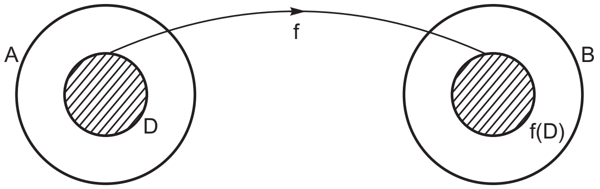


Fig. 1.IV. The image *f*(*D*)

It is clear that $f(A) \subset B$. If $f(A) = B$, then we call f a mapping of A onto B. The mapping f of A into B is called one-to-one mapping if two different elements in A have different images under f. In other words, f is one-to-one mapping of A into B if whenever $x_1 \neq x_2, x_1 \in A, x_2 \in A$ implies that $f(x_1) \neq f(x_2)$.

If $f : A \to B$ is both onto and one-to-one, then we can define the inverse **mapping.** $f^{-1} : B \to A$ in the following way :

$$f^{-1}(y) = x \text{ if and only if } f(x) = y$$

for every $y \in B$ and for every $x \in A$.

Let $f : A \to B$ be a map. Suppose now that G is a subset of B. The inverse image $f^{-1}(G)$ is the subset of A defined by

$$f^{-1}(G) = \{x : f(x) \in G\}.$$

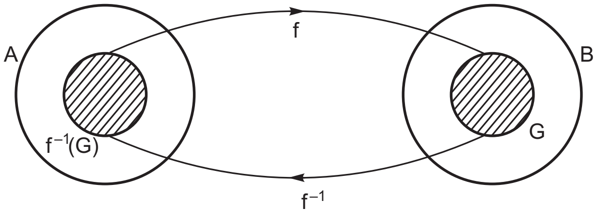


Fig. 1.V. The inverse image $f^{-1}(G)$

We list below certain properties of sets that are preserved under the direct image and inverse image, respectively. The reader should convince himself of the validity of these properties.

Properties: Let $f : A \to B$ be a map. Suppose that D and E are subsets of A.

(*a*) If $D \subset E$, then $f(D) \subset f(E)$;

(*b*) $f(D \cap E) \subset f(D) \cap f(E)$;

(*c*) $f(D \cup E) = f(D) \cup f(E)$;

(*d*) $f(D \setminus E) \subset f(D)$.

Properties: Let $f : A \to B$ be a map. Suppose that G and H are subsets of B.

(*a*) If $G \subset H$, then $f^{-1}(G) \subset f^{-1}(H)$;

(*b*) $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$;

(*c*) $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$;

(*d*) $f^{-1}(G - H) = f^{-1}(G) - f^{-1}(H)$.

Observe that the inverse image is better behaved than the direct image.

Let $f : A \to B$ and $g : B \to C$ be any two functions. The **composition** $g \, o \, f$ is the function from $A \to C$ defined as follows:

$(x, z) \in g \, o \, f$ if and only if for some y, $(x, y) \in f$ and $(y, z) \in g$.

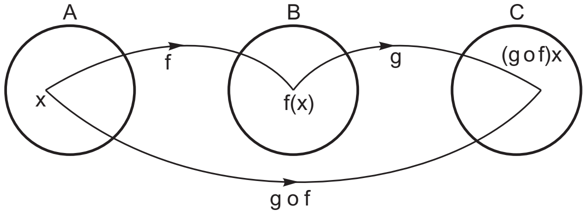


Fig. 1.VI. Composition of functions

Note that if f and g are one-to-one, then $g \circ f$ is also one-to-one and its inverse function is

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Let $f : A \to B$ be a function and let E be a subset of A. We call the set of elements $(x, f(x))$ with $x \in E$ the **restriction** of f to E and write

$$f \mid E : E \to B.$$

Clearly, if f is one-to-one, so is $f \mid E$.

1.3 COUNTABLE SETS

Definition: Let A and B be any two sets. If the function $f : A \to B$ is one-to-one and onto, then we say that f is a **one-to-one correspondence** between A and B.

Example: Let A denotes the set of positive odd integers and B the set of positive even integers. Define $f : A \to B$ by

$$f(n) = n + 1.$$

Then f is one-to-one correspondence between A and B.

Whenever there exists a one-to-one correspondence from set A to set B, we say that A and B have the same **cardinal number,** or briefly, that A and B are equivalent, and write

$$A \approx B.$$

This relation has the following properties and can be easily verified.

(*a*) reflexive : $A \approx A$ for any set A.

(*b*) symmetric : if $A \approx B$, then $B \approx A$.

(*c*) transitive : if $A \approx B$ and $B \approx C$, then $A \approx C$.

A set A is called finite if it is empty or if there is a natural number n such that A is equivalent to the set $\{1, 2, 3, ..., n\}$. A set A is called infinite if A is not finite. The most basic infinite set is the set of all positive integers:

$$I = \{1, 2, 3, .., n, n + 1, ...\}.$$

Definition: We say that a set A is countable if $A \approx I$.

Example: Let A be the set of all integers.

Define $f: I \to A$ by

$$f(n) = \begin{cases} \dfrac{n}{2} & (n \text{ even}); \\ -\dfrac{n-1}{2} & (n \text{ odd}). \end{cases}$$

The function is one-to-one correspondence from A to I. The sets A and I are arranged as follows:

$$\begin{array}{ccccccc} A: & 0, & 1, & -1, & 2, & -2, & \ldots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ I: & 1, & 2, & 3, & 4, & 5, & \ldots \end{array}$$

We say that a set A is uncountable if A is neither finite nor countable.

Definition: Let I be the set of all positive integers. A sequence in a set A is a function $f: I \to A$.

In other words, a sequence assigns to each $n \in I$, a uniquely determined element of A.

If $f(n) = x_n$, for $n \in I$, we usually denote the sequence f by the symbol

$$\{x_n\} \text{ or } x_1, x_2, x_3, \ldots, x_n, \ldots .$$

If $x_n \in A$ for all $n \in I$, then we say that x_n is a sequence in A.

We need to define unions and intersections of large class of sets. Let $\{A_i\}$ be a collection of sets, where i runs through some index set I. We define their **union** and **intersection** as follows:

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for at least one } i \in I\}$$

and
$$\bigcap_{i \in I} A_i = \{x : x \in A_i \text{ for every } i \in I\}$$

If I is the set of all positive integers, then their union and intersection are often written in the form

$$\bigcup_{i=1}^{\infty} A_i \quad \text{and} \quad \bigcap_{i=1}^{\infty} A_i.$$

Note that De Morgan's laws can be generalized for an arbitrary collection of sets in the following way:

(*a*) $A - \bigcup_{i \in I} A_i = \bigcap_{i \in I} (A - A_i)$;

(*b*) $A - \bigcap_{i \in I} A_i = \bigcup_{i \in I} (A - A_i)$.

We shall now establish two useful properties of countable sets.

Theorem 1. *Every infinite subset F of a countable set E is countable.*

Proof: Let the element $x \in E$ be arranged in a sequence $x_1, x_2, x_3, \ldots, x_n, \ldots$ of distinct elements. We construct a sequence $n_1, n_2, n_3, \ldots$ in the following way.

Let n_1 be the smallest positive integer such that $x_{n_1} \in F$. Having chosen n_1, let n_2 be the smallest integer greater than n_1 such that $x_{n_2} \in F$. In this way, having chosen $n_1, n_2, \ldots, n_{k-1}$ $(k = 2, 3, 4, \ldots)$, let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in F$.

Set $$f(k) = x_{n_k} \quad (k = 1, 2, 3, \ldots).$$

We thus see that there exists a one-to-one correspondence between F and the set of all positive integers.

Theorem 2. *The union of a countable collection of countable sets is countable.*

Proof: Let $\{A_i\}$ be a sequence of countable sets, and let

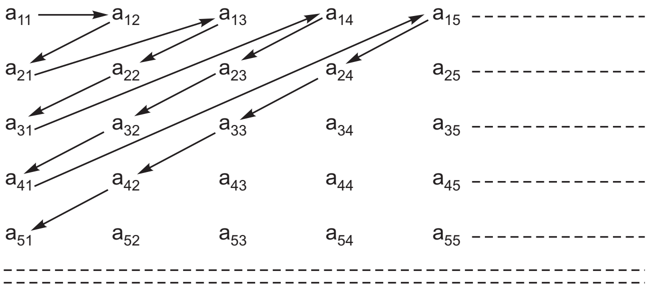
$$E = \bigcup_{i=1}^{\infty} A_i.$$

Then we have to prove that E is countable.

Let $\{A_1, A_2, A_3, \ldots\}$ be arranged in a sequence $\{a_{ij}\}$, $i, j = 1, 2, 3, \ldots$. Let all the elements a_{ij} be written in the following order.

Note that we have listed the elements of A_1 in the first row, the elements of A_2 in the second row, and so on. Write all these elements (as indicated by the arrows) in the form of a sequence

$$a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, \ldots$$



Observe that a_{pq} precedes a_{mn} if $p + q < m + n$, or, in case $p + q = m + n$, if $p < m$.

Hence, it follows from Theorem 1 that E is countable.

1.4 FIELDS

In order to introduce the notion of a "field" we shall follow a convention that is familiar to the reader from elementary courses in modern algebra. By a **binary operation** in a set F, we mean a function

$$h : F \times F \to F.$$

In general, $h(a, b)$ denotes the value of the binary operation h at the point (a, b) in $F \times F$, but we shall use symbols such as $a + b$ or $a \cdot b$.

Definition: A set F is called a **field** if there are two binary operations + and · such that the following properties are satisfied.

If a, b, c belong to F, then

(*i*) $a + b = b + a$;

(*ii*) $(a + b) + c = a + (b + c)$;

(*iii*) $a{\cdot}b = b{\cdot}a$;

(*iv*) $(a{\cdot}b){\cdot}c = a{\cdot}(b{\cdot}c)$;

(*v*) $a{\cdot}(b + c) = (a{\cdot}b) + (a{\cdot}c)$ and $(b + c){\cdot}a = (b{\cdot}a) + (c{\cdot}a)$;

(*vi*) there exists a unique element θ in F such that $\theta + a = a$ and $a + \theta = a$;

(*vii*) for each element a in F there is an element $\overline{a}$ in F such that $a + \overline{a} = \theta$ and $\overline{a} + a = \theta$.

(*viii*) there exists a unique element $e \neq \theta$ in F such that $e{\cdot}a = a$, $a{\cdot}e = a$;

(*ix*) for each element $a \neq \theta$ in F there is an element a' in F such that $a \cdot a' = e$, and $a' \cdot a = e$.

The element θ is called the **zero element** of F and the element e is called the **identity** or **unit element** of F.

We assume that the reader is familiar with the "algebraic" structure of the real number system. Before we discuss the "algebraic" structure of the complex number system, we shall give two examples of fields.

Examples: (*i*) Consider the system R of real numbers. Two of the binary operations in R are addition and multiplication. Note that the familiar operations subtraction and division are defined in terms of addition and multiplication respectively. Here $\theta = 0$ and $e = 1$ are the additive and multiplicative identities of R. For every $a \in R$, there exists $\overline{a} = (-1)\, a$ in R such that $a + (-1)\, a = 0$ and $(-1)\, a + a = 0$. For every $a \neq 0$ in R, there exists a number $a' = 1/a$ such that $a'a = 1$ and $aa' = 1$. We conclude that the set R of real numbers forms a field.

(*ii*) Let Q denotes the set of rational numbers; that is, real numbers of the form p/q where p and q are integers and $q \neq 0$. In this case also, $\theta = 0$ and $e = 1$. It is readily checked that the set Q of all rational numbers forms a field.

1.5 COMPLEX NUMBERS

Let R be the set of real numbers. A **complex number** z is defined as an ordered pair of real numbers:

$$z = (a, b) \text{ where } a \in R,\ b \in R.$$

The real numbers a and b are called the real and the imaginary part of z, respectively. In symbols, $a = \text{Re}\, z$, $b = \text{Im}\, z$.

Two complex numbers $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$ are said to be equal if and only if $a_1 = a_2$ and $b_1 = b_2$. We denote by C the set of all complex numbers.

The operations of addition and multiplication in C are defined as follows:

$$z_1 + z_2 = (a_1 + a_2, b_1 + b_2);$$

$$z_1 z_2 = (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1);$$

$$\lambda z = \lambda\,(a, b) = (\lambda a, \lambda b) \text{ where } \lambda \in R.$$

It can be easily checked that with these definitions C satisfies the **associative, commutative** and **distributive** laws for addition and multiplication.

The complex numbers (0, 0) and (1, 0) are the additive and multiplicative identities of C. The operations of subtraction and division in C are defined as

$$z_1 - z_2 = (a_1 - a_2, b_1 - b_2);$$

$$\frac{z_1}{z_2} = \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}, \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right) \text{ if}$$

$$z_2 = (a_2, b_2) \neq (0, 0).$$

It is readily seen that the set C of all complex numbers with respect to the operations described above is a field.

Note that for $a, b \in R$, the following properties hold.

(*i*) $(a, 0) + (b, 0) = (a + b, 0)$;

(*ii*) $(a, 0)\,(b, 0) = (ab, 0)$;

(*iii*) $\dfrac{(a, 0)}{(b, 0)} = \left(\dfrac{a}{b}, 0\right)$ if $b \neq 0$.

This shows that we can identify $(a, 0)$ with a and hence may consider R as a subset of C. It is now desirable to show that the notation (a, b) is equivalent to the more customary $a + ib$. If we introduce

$$i = (0, 1)$$

and identify $(a, 0)$ and $(b, 0)$ with a and b, respectively, then

$$a + ib = (a, 0) + (0, 1)\,(b, 0) = (a, 0) + (0, b)$$

$$= (a, b).$$

Note that

$$i^2 = (0, 1)\,(0, 1) = (-1, 0) = -1.$$

It may be remarked here that the equation $x^2 + 1 = 0$, which has no solution in R, becomes solvable in C. The roots of the equation are $\pm i$.

Before we conclude this section, it is important to mention that the field of complex numbers is not an ordered field like that of real numbers and rational numbers. We shall not discuss it here and readers interested in this topic may consult Birkhoff and Maclane [5], or any other good book on modern algebra.

1.6 THE COMPLEX PLANE

It is customary to use the letter *R* to denote the set of all real numbers as well as the real line, and identify the real numbers with the points on the real line. For the geometric representation of the complex numbers, it is necessary to introduce first the coordinate plane.

Take two mutually perpendicular axes, the *x*-axis and the *y*-axis, as shown in Fig. 1.VII. Let *P* be a point in the plane. Project *P* perpendicularly onto points P_x and P_y on the *x* and *y* axes, respectively. Let *x* and *y* be the coordinates of P_x and P_y on the *x* and *y* axes, respectively. Then it can be easily seen that there is a one-to-one correspondence between the ordered pairs (x, y) of real numbers and point *P* in the plane. The set of ordered pairs (x, y) with $x \in R$ and $y \in R$ is called the **coordinate plane** and is denoted by $R \times R$, or R^2.

It follows from the definition of complex numbers that each $z = x + iy$ in *C* can be identified with the unique point (x, y) in the plane R^2. Thus with each complex number $z = x + iy$, we associate the point (x, y) in the plane, and vice-versa, each point of that plane corresponds to one and only one complex number. We call this plane the **complex plane** and denote by *C* itself. In the complex plane *C*, the *x*-axis of the coordinate system is called the **real axis** and the *y*-axis is called the **imaginary axis.**

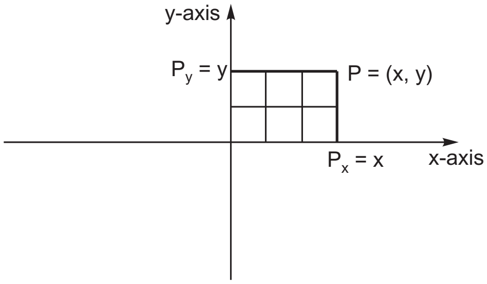


Fig. 1.VII

Let $z = x + iy$ be a complex number. We introduce the polar coordinate (r, θ):

$$x = r\cos\theta,\ y = r\sin\theta.$$

The positive quantity *r* is called the **absolute value** of *z* and is denoted by $r = |z|$. The angle θ is the angle between the positive real axis and the line segment from 0 to *z*. We illustrate this in Fig. 1.VIII.

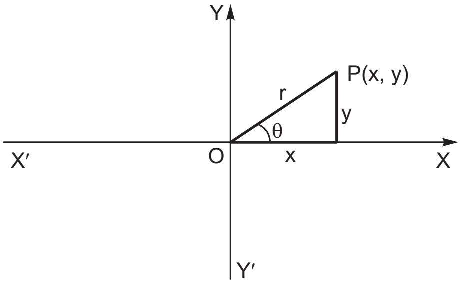


Fig. 1.VIII

The angle θ is called the **argument** of z and is denoted by $\theta = \arg z$. If θ is one of the value of arg z, then

$$\arg z = \theta + 2k\pi,\ k = 0, \pm 1, \pm 2, \ldots$$

The value of θ which lies in the interval

$$-\pi < \theta \leq \pi$$

is called the **principal value** of the argument of z.

Note that the arg z is not unique and is infinite valued.

We introduce the notation

$$\text{cis}\ \theta = \cos\theta + i\sin\theta.$$

Let $$z_1 = r_1 \text{ cis } \theta_1,\ z_2 = r_2 \text{ cis } \theta_2.$$

Then $$\begin{aligned} z_1z_2 &= r_1r_2 \text{ cis } \theta_1 \text{ cis } \theta_2 \\ &= r_1r_2\,[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \\ &\qquad + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]. \end{aligned}$$

By applying the familiar addition formulas for the sine and cosine, we get

$$z_1z_2 = r_1r_2 \text{ cis } (\theta_1 + \theta_2).$$

Thus $$|z_1z_2| = |z_1|\,|z_2|$$

and $$\arg(z_1z_2) = \arg z_1 + \arg z_2 \text{ (upto multiples of } 2\pi).$$

We write $$z_k = r_k \text{ cis } \theta_k,\ 1 \leq k \leq n.$$

It follows by induction that

$$z_1z_2 \ldots z_n = r_1r_2 \ldots r_n \text{ cis } (\theta_1 + \theta_2 + \ldots + \theta_n).$$

In particular,

(1) $$z^n = r^n \text{ cis } (n\theta) \text{ for every integer } n \geq 0.$$

The quotient of two complex numbers can be handled in a similar way. If $z \neq 0$,

$$z\,[r^{-1} \text{ cis } (-\theta)] = 1;$$

so that (1) holds for all integers n, positive, negative and zero, if $z \neq 0$. In the special case $r = 1$ of (1) we get de Moivre's formula:

(2) $$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

1.7 THE ROOTS OF COMPLEX NUMBERS

We can use **de Moivre's formula** to find the nth roots of a complex number $z = r \text{ cis } \theta$. Write $z = w^n$ where $w = R \text{ cis } \psi$

If follows from de Moivre's formula that

$$z = w^n = R^n(\cos n\psi + i\sin n\psi).$$

Thus

(3) $$r(\cos\theta + i\sin\theta) = R^n(\cos n\psi + i\sin n\psi).$$

By equating the absolute values on both sides in (3), we get

$$R^n = r \text{ i.e. } R = \sqrt[n]{r}$$

where the root is real positive and thus is uniquely determined.

By equating the arguments in (3), we obtain

$$n\psi = \theta + 2k\pi$$

i.e.

$$\psi = \frac{\theta}{n} + \frac{2k\pi}{n}$$

where k is an integer.

Consequently, $\sqrt[n]{z}$, for $z \neq 0$, has the n distinct values.

(4) $$\sqrt[n]{z} = \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right), \ k = 0, 1, \ldots, n-1.$$

The complex number $x - iy$, denoted by $\bar{z}$, is called the **conjugate** of

$$z = x + iy.$$

Note that

$$\sqrt{z\,\bar{z}} = \sqrt{x^2 + y^2} = |z|.$$

If $z \neq 0$, then

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}.$$

The following basic properties of absolute values and conjugates can be easily verified:

(*i*) $\operatorname{Re} z = \frac{1}{2}(z + \bar{z}), \operatorname{Im} z = \frac{1}{2i}(z - \bar{z}).$

(*ii*) $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2.$

(*iii*) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, |z_2| > 0.$

(*iv*) $|\bar{z}| = |z|.$

Note also that $|z_1 - z_2|$ is exactly the distance between z_1 and z_2. The fundamental properties of the distance function are:

If $z, z_1, z_2 \in C$, then

(*a*) $|z| \geq 0, |z| = 0$ if and only if $z = 0$;

(*b*) $|z_1 z_2| = |z_1||z_2|$;

(*c*) $|z_1 + z_2| \leq |z_1| + |z_2|$;

(*d*) $|z_1 - z_2| \geq |z_1| - |z_2|.$

We prove only the statement (*c*), which is called the **triangle inequality.**

Observe that for any $z_1 \in C$,

$$-|z_1| \le \text{Re}\, z_1 \le |z_1|\,;$$
$$-|z_1| \le \text{Im}\, z_1 \le |z_1|.$$

Hence

$$\text{Re}(z_1\bar{z}_2) \le |z_1\bar{z}_2| = |z_1|\,|z_2|.$$

Thus

$$\begin{aligned} |z_1 + z_2|^2 &= |z_1|^2 + 2\,\text{Re}\,(z_1\bar{z}_2) + |z_2|^2 \\ &\le |z_1|^2 + 2\,|z_1|\,|z_2| + |z_2|^2 \\ &= (|z_1| + |z_2|)^2, \end{aligned}$$

from which (*c*) follows.

The triangle inequality can easily be generalized *i.e.*

(*e*) $$\left|\sum_{i=1}^{n} z_i\right| \le \sum_{i=1}^{n} |z_i|$$

The equality in (*e*) holds if and only if

$$\arg z_1 = \arg z_2 = \ldots = \arg z_n.$$

A complex number $z = x + iy$ can be represented by the directed line segment *OP* joining the origin *O* to the point *P* (*x*, *y*). We often call *OP* the position vector of *P*.

Let OP_1 and OP_2 represent the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, respectively. We complete the parallelogram $OP_1\ PP_2$ whose side OP_1 and OP_2 correspond to z_1 and z_2. It is readily verified that addition of complex numbers corresponds, in the complex plane, to a **parallelogram law for addition of vectors.** This fact is illustrated in Fig. 1.IX.

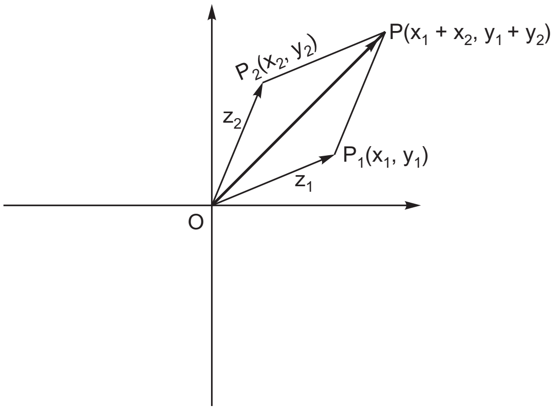


Fig. 1.IX

1.8 STEREOGRAPHIC PROJECTION

Let C be the complex plane. Consider a unit sphere S which is centered at $z = 0$ of the complex plane C. Suppose that the sphere S is cut by the complex plane C along its equator, which coincides with the unit circle centred at $z = 0$, as indicated in Fig. 1.X. The diameter NN' is perpendicular to C. We call points N and N' the north and south poles of the sphere S. A point P in C is mapped onto the point P' on the sphere S by joining N to P by a line. The line NP intersects the sphere S at P'. Points inside the unit circle are mapped onto the lower hemisphere, and points outside the unit circle are mapped onto the upper hemisphere. The origin $z = 0$, is mapped onto the south pole N' of the sphere S. We now put the north pole N in correspondence with the "point at infinity" of the complex plane C.

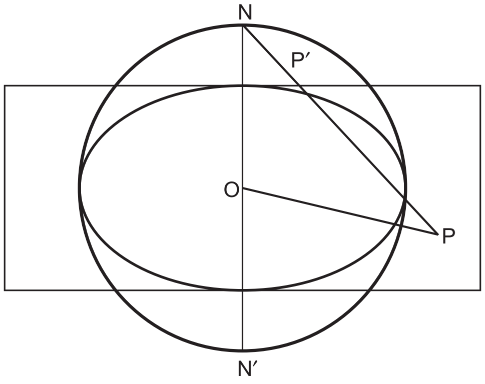


Fig. 1.X

The method described above for mapping the plane onto the sphere is called the **Stereographic Projection.** The set of all points of the complex plane including the point at infinity is called the **extended complex plane.** The sphere is often called the **Riemann Sphere.** This is the manner in which a one-to-one correspondence is established between the sphere and the extended complex plane.

1.9 SPHERICAL REPRESENTATION

We denote the extended plane by $C_\infty = C \cup \{\infty\}$. Let $P = x + iy$ and let $P' = (x_1, x_2, x_3)$ be the corresponding point on S. The line in R^3 through P and N is given by

(5) $$[\{(1-u)\,x,\ (1-u)\,y,\ u\} : -\infty < u < \infty]$$

In order to find the coordinates of P' we have to find the value of u at which this line intersects S. If u is the value, then

$$\begin{aligned} 1 &= (1-u)^2 x^2 + (1-u)^2 y^2 + u^2 \\ &= (1-u)^2 \,|\,P\,|^2 + u^2. \end{aligned}$$

It follows that

$$1 - u^2 = (1-u)^2 \,|\,P\,|^2$$

Since $u \neq 1$ $(P \neq \infty)$ we get

$$u = \frac{|P|^2 - 1}{|P|^2 + 1}$$

Thus $$x_1 = \frac{2x}{|P|^2 + 1},\ x_2 = \frac{2y}{|P|^2 + 1},\ x_3 = \frac{|P|^2 - 1}{|P|^2 + 1}$$

We can write the above expression as

(6) $$x_1 = \frac{P + \overline{P}}{|P|^2 + 1},\ x_2 = \frac{P - \overline{P}}{|P|^2 + 1},\ x_3 = \frac{|P|^2 - 1}{|P|^2 + 1}$$

Now we have to find P when the point P' $(P' \neq N)$ is given. Setting $u = x_3$ and using (5) we get

$$P = \frac{x_1 + ix_2}{1 - x_3}$$

Let P, Q be in C_∞. Define the distance from P to Q, $d(P, Q)$ to be the distance between the corresponding points P', Q' in R^3. If $P' = (x_1, x_2, x_3)$ and $Q' = (x'_1, x'_2, x'_3)$, then

(7) $$d(P, Q) = \{(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2\}^{1/2}$$

Since P' and Q' are on S, it follows from (7) that

$$[d(P, Q)]^2 = 2 - 2\,\{x_1 x'_1 + x_2 x'_2 + x_3 x'_3\}.$$

Using (6) we find that

$$d(P, Q) = \frac{2|P - Q|}{\{(1 + |P|^2)(1 + |Q|^2)\}^{1/2}},\ (P, Q \in C)$$

Proceeding in a similar way we can obtain

$$d(P, \infty) = \frac{2}{(1 + |P|^2)^{1/2}},\ (P \in C)$$

Thus, the correspondence between points of S and C_∞ gives the spherical representation of the extended plane.

EXERCISES

1. The symmetric difference between two sets A and B, denoted by $A \Delta B$, is defined by

$$A \Delta B = (A - B) \cup (B - A).$$

(*i*) Show that Δ satisfies associative law and commutative law.

(*ii*) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$.

(*iii*) $A \Delta \phi = A$, $A \Delta A = \phi$.

2. Let $f : A \to B$; let S and T be subsets of A, and D and E be subsets of B, Prove the following;

(*i*) $f(f^{-1}(D)) \subset D$, and give an example in which

$$f(f^{-1}(D)) \neq D.$$

(*ii*) $f(S) - f(T) \subset f(S - T)$, and give an example in which

$$f(S) - f(T) \neq f(S - T).$$

(*iii*) $f^{-1}(D) - f^{-1}(E) = f^{-1}(D - E)$.

3. Let X be any set. The mapping f defined on X by

$$f(x) = x \text{ for every } x \in X,$$

is called the identity mapping on X. We will denote this mapping by I_X. Note that such a mapping keeps every element of X fixed.

Let $f : X \to Y$ be one-to-one and onto, and let f^{-1} be its inverse. Prove that

$$f \, o \, f^{-1} = I_Y.$$

Explain how does I_Y differ from I_X.

4. Prove that there exists a one-to-one correspondence between the set of positive integers and the set of all positive rational numbers.

5. Prove that the set of all real numbers between zero and one is not countable.

6. Prove that the set of all rational numbers is countable.

7. Prove that the set of all points whose coordinates are both rational is countable.

8. If A and B are countable, then prove that $A \times B$ is countable.

9. Express the following complex numbers in the form $a + ib$ where a, b are real numbers

(*i*) $\dfrac{1}{-1 + 2i}$ (*ii*) $i(2 + 3i)^4$

(*iii*) $(1 + \pi i)(i + \pi)$ (*iv*) $\dfrac{1+i}{1-i} - \dfrac{1-i}{1+i}$

(*v*) $\dfrac{(1+i)^3}{(2+i)(1+2i)}$

10. Express the following complex numbers in polar form

(*i*) $1 + i\sqrt{3}$ (*ii*) $-2\sqrt{3} - 2i$

(*iii*) $-4i$ (*iv*) $-1 - i$

(*v*) $1 - i\sqrt{2}$ (*vi*) $2\sqrt{2} + i2\sqrt{2}$

(*vii*) $\sqrt{2}i$ (*viii*) -4

(*ix*) $\sqrt{3}/2 - \dfrac{3}{2}i$

11. If $z = \dfrac{(3+4i)(12-5i)}{1+i}$, compute

$$|z|, \text{Re } z \text{ and Im } z.$$

12. Compute all the roots of the equation $z^3 = 1$. If ω is one of the complex cube roots of unity show that the other is ω^2. Verify that $1 + \omega + \omega^2 = 0$.

13. Compute all roots and plot them in the complex plane:

(*i*) $(-1)^{1/4}$ (*ii*) $(-1 + i)^{1/3}$

(*iii*) $i^{2/3}$ (*iv*) $(-16i)^{1/4}$

(*v*) $(-27i)^{1/6}$ (*vi*) $(1-i)^{-1/2}$

(*vii*) $8^{1/3}$ (*viii*) $(-1-i)^{4/5}$

(*ix*) $(4\sqrt{2}+i4\sqrt{2})^{1/2}$

14. If $a + ib$ is a root of

$$a_0z^n + a_1z^{n-1} + \dots + a_n = 0 \text{ where } a_n \neq 0,$$

$a_1, a_2, \dots, a_n$, a and b are real, prove that $a - ib$ is also a root.

15. If $x_1 + iy_1$ is one complex root of the equation $z^5 = 1$ show that

$$4x_1\,(y_1^4 - x_1^4) = 1.$$

16. If z is a complex number such that $z\,\bar{z} = 1$, compute

$$|\,1+z\,|^2 + |\,1-z\,|^2.$$

17. If $w = \sqrt{z_1z_2}$, show that

$$|\,z_1\,| + |\,z_2\,| = \left|\,\frac{z_1+z_2}{2} + w\,\right| + \left|\,\frac{z_1+z_2}{2} - w\,\right|.$$

18. If z_1, z_2 are complex numbers, show that

(*a*) $|\,z_1 - z_2\,|^2 + |\,z_1 + z_2|^2 = 2\,|\,z_1|^2 + 2\,|\,z_2|^2$;

(*b*) $|\,|\,z_1\,| - |\,z_2\,|\,| \leq |\,z_1 - z_2\,|$.

19. Show that three points z_1, z_2, z_3 in the complex plane are collinear iff there exist real numbers a, b, c not all zero, such that $a + b + c = 0$ and $az_1 + bz_2 + cz_3 = 0$.

20. Show that triangles whose vertices are the points z_1, z_2, z_3 and z_4, z_5, z_6, respectively, are similar iff

$$\begin{vmatrix} z_1 & z_4 & 1 \\ z_2 & z_5 & 1 \\ z_3 & z_6 & 1 \end{vmatrix} = 0.$$

21. Show that if the points z_1, z_2, z_3 are the vertices of an equilateral triangle, then

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$

Is the converse true ?

22. If z_1, z_2, z_3, z_4 are the position vectors of the vertices for quadrilateral *ABCD*, show that *ABCD* is a parallelogram iff

$$z_1 - z_2 + z_3 - z_4 = 0.$$

23. Show that the locus of z defined by the equation $|\,z - w_1\,| = |\,z - w_2|$ is the perpendicular bisector of the line *AB*, where *A* and *B* are the points in the complex plane which represent the complex numbers w_1 and w_2 respectively. Deduce that

$$\arg\frac{2z - w_1 - w_2}{w_1 - w_2} = \pm\,\pi/2.$$

24. If z_1, z_2, z_3 and z_4 are four distinct points of the complex plane, show that they lie on a circle or a straight line iff

$$\frac{(z_3 - z_1)\,(z_4 - z_2)}{(z_3 - z_2)\,(z_4 - z_1)}$$

is real.

25. Describe geometrically the set of points z satisfying the following conditions.

(*a*) $|z - i| = 2$

(*b*) $|z + 2i| + |z - 2i| = 6$

(*c*) $|z - 3| - |z + 3| = 4$

(*d*) $z(\bar{z} + 2) = 3$

(*e*) $|z + 3i| \leq 1$

(*f*) $\text{Re}\, z \geq 0$

(*g*) $\text{Im}\, z \geq 0$

(*h*) $\text{Re}\, z + \text{Im}\, z = 0$

(*i*) $\text{Im}\, z^2 = 4$

(*j*) $|\text{Re}\, z| + |\text{Im}\, z| = 1.$

26. Determine the locus of the points z which satisfy

$$\arg\left(\frac{z - a}{z - b}\right) = \pi/2.$$

2

METRIC SPACES

2.1 DEFINITION

A **metric space** is a pair (X, d) where X is a set and d is a mapping from $X \times X$ into R which satisfies the following conditions:

(*i*) $d(x, y) \geq 0$;

(*ii*) $d(x, y) = 0$ iff $x = y$;

(*iii*) $d(x, y) = d(y, x)$;

(*iv*) $d(x, z) \leq d(x, y) + d(y, z)$

for $x, y, z \in X$.

Examples: (*a*) Let $X = R$ or C and define d by

$$d(z_1, z_2) = |z_1 - z_2|.$$

(*b*) Let $X = R^n$ and define d by

$$d(x, y) = |x - y| = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2},$$

where $x = (x_1, x_2, ..., x_n)$ and

$y = (y_1, y_2, ..., y_n)$; $x_i, y_i \in R$, $i = 1, 2, ..., n$.

(*c*) Let X be a non-empty set and define d by

$$d(x, y) = \begin{cases} 0 \text{ if } x = y; \\ 1 \text{ if } x \neq y. \end{cases}$$

This metric space is called a **discrete metric space.**

Open Ball: Let (X, d) be a metric space. If $a \in X$ and $r > 0$, then the set

$$\{x : x \in X, d(x, a) < r\},$$

denoted by $B_r(a)$, is called the open ball with centre a and radius r.

The open ball $B_r(a)$ on R is the bounded open interval $(a - r, a + r)$ with mid-point a and total length $2r$. The open ball $B_r(a)$ on C is the set

$$\{z \in C : |z - a| < r\}.$$

Figure 2.I illustrates an open ball in the complex plane C. The open ball on C is also called the **open disc**.

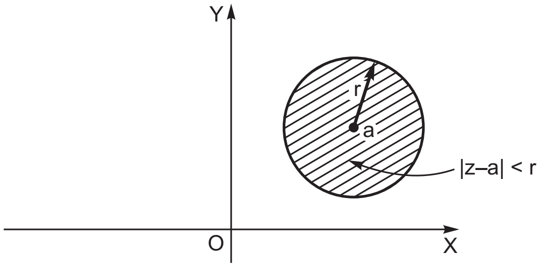


Fig. 2.I

Open Sets: Let (X, d) be a metric space. A set $G \subset X$ is open if for each $x \in G$ there is an $r > 0$ such that $B_r(x) \subset G$.

Examples: (*a*) The set $G = \{z \in C : a < \text{Re}\, z < b\}$ is open.

(*b*) The set $S = \{z \in C : \text{Re}\, z < 0\} \cup \{0\}$ is not open.

Note that the empty set ϕ and the full space X are open sets. Observe that in any metric space (X, d), each open ball is an open set.

Closed Sets: Let (X, d) be a metric space. The set $G \subset X$ is said to be closed if the complement $X - G$ is open.

We need the following characterization of open sets in terms of open balls.

Lemma 2.1

Let (X, d) be a metric space. A subset E of X is open if and only if it is a union of open balls.

Proof: Suppose that E is the union of a collection F of open balls. If F is empty, then E is also empty and hence it is open. Suppose that F is non-empty. E is also non-empty. Let $x \in E$. Since E is the union of the open balls in F, $x \in B_r(x_0)$ which is contained in F. Since each open ball is an open set, there exist an open ball $B_{r_1}(x) \subset B_r(x_0)$. Hence,

$$B_{r_1}(x) \subset E, \text{ and so, } E \text{ is open.}$$

To prove the converse, assume that E is open. Since E is open, each of its points is the centre of an open ball contained in E. Hence E is the union of all the open balls contained in E.

Theorem 2.1. *Let (X, d) be a metric space. Then*

(*i*) *the sets X and ϕ are open;*

(*ii*) *the union of any number of open sets in X is open;*

(*iii*) *the intersection of a finite number of open sets in X is open.*

Proof: (*i*) Since every open ball centred on each of its points is contained in X, hence X is open.

ϕ is clearly open, since the requirement of the definition of open set is automatically satisfied.

To prove (*ii*), let $\{H_i : i \in I\}$, where I is any index set, be a class of open sets in X. Let $H = \underset{i \in I}{\cup} H_i$. If $\{H_i\}$ is empty, then H is empty and hence H is open.

Assume that $\{H_i\}$ is non-empty. Since each H_i is a union, of open balls, H is the union of all the open balls. Hence, by Lemma 2.1, H is open.

To prove (*iii*), let $\{G_i : i = 1, 2, ..., n\}$ be a finite number of open sets in X. Let $G = \underset{i \in I}{\cap} G_i$. If $\{G_i\}$ is empty, then G becomes the whole space X and hence G is open.

Assume that G is non-empty. Let $x \in G$. Since x is in each G_i, and each G_i is open. Thus for each i there is a positive real number r_i such that

$$B_{r_i}(x) \subset G_i.$$

Set $r = \min\{r_1, r_2, ..., r_n\}$ and note that

$$B_r(x) \subset \bigcap_{i=1}^{n} G_i. \quad \text{Hence} \quad \bigcap_{i=1}^{n} G_i \text{ is open.}$$

Theorem 2.2. *Let (X, d) be a metric space. Then*

(*i*) *the sets X and* ϕ *are closed;*

(*ii*) *the intersection of any number of closed sets in X is closed;*

(*iii*) *the union of a finite number of closed sets in X is closed.*

Proof: (*i*) The result follows from Theorem 2.1(*i*) and the fact that $X^c = \phi$, $\phi^c = X$, where "*c*" stands for the complement. To prove (*ii*), let $\{F_i : i \in I\}$ be a class of closed sets. Then by De Morgan's law and Theorem 2.1 we see that

$$\left(\bigcap_{i \in I} F_i\right)^c = \bigcup_{i \in I} F_i^c \text{ is open.}$$

Thus $$\bigcap_{i \in I} F_i \text{ is closed.}$$

The proof of (*iii*) is similar to that of (*ii*).

Note that a set which is not closed is not necessarily open, and vice-versa.

Interior: Let (X, d) be a metric space and S a subset of X. A point $x \in S$ is called an **interior point** of S if there exist an open ball $B_r(x)$ such that $B_r(x) \subset S$. The interior of S, denoted by S^0, is the set of all its interior points.

Observe that $S^0 \subset S$. It is easy to verify that S is open iff $S = S^0$.

Closure: Let (X, d) be a metric space and S a subset of X. A point $x \in X$ is called a **closure point** of S if every open ball centred on x contains at least one point of S. In other words, a point $x \in X$ is a closure point of S if

$$B_r(x) \cap S \neq \phi \text{ for all } r > 0.$$

The closure of S, denoted by $\overline{S}$, is the set of all its closure points.

Observe that $S \subset \overline{S}$.

Closed Ball: Let (X, d) be a metric space. Let $a \in X$ and let $r > 0$. Then the set $\{x \in X : d(x, a) \leq r\}$ is called the closed ball with centre a and radius r.

It can be easily proved that in a metric space (X, d), each closed ball is a closed set.

For, let $d(x, a) > r$, and let $r_1 = d(x, a) - r > 0$. If $d(y, x) < r_1$, then $d(a, y) \geq d(a, x) - d(y, x) > d(a, x) - r_1 = d(a, x) - [d(a, x) - r] = r$. This shows the S^c is open, and thus, S is closed.

In general, it is not true that in every metric space the closure of every open ball of radius r is the closed ball of radius r.

Here is an example. Let (X, d) be a discrete metric space (*i.e.*, $d(x, y) = 1$ if $x \neq y$, $d(x, x) = 0$). Then

$$B_1(a) = \{x \in X : d(x, a) < 1\} = \{a\};$$
$$\{x \in X : d(x, a) \leq 1\} = X;$$
$$\overline{B_1(a)} = \{a\}.$$

Limit Point: Let (X, d) be a metric space and S a subset of X. A point $x \in X$ is called a limit point (an **accumulation point**) of S if each open ball $B_r(x)$ contains at least one point of S different from x. In other words, a point $x \in X$ is a limit point of S if

$$B_r(x) \cap (S - \{x\}) \neq \phi \text{ for each } r > 0.$$

It is clear that every limit point of a set must be a closure point of that set.

The set of all limit points of S is called the **derived set** of S, and is denoted by S'.

Note that $\overline{S} = S \cup S'$. Also, note that S is closed iff it contains all its limit points.

Examples: (*a*) Let $X = R$ and $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$.

0 is *a*–the only limit point of S.

(*b*) Let $X = C$ and $S = [0, 1) \cup \{i\}$.

The set of all points of $[0, 1]$ is a limit point of S but i is not.

Boundary: Let (X, d) be a metric space and S a subset of X. A point $x \in X$ is called a **boundary point** of S if every open ball $B_r(x)$ intersects both S and S^c. In other words, a point $x \in X$ is a boundary point of S if

$$B_r(x) \cap S \neq \phi \text{ and } B_r(x) \cap S^c \neq \phi$$

for all $r > 0$.

The boundary of S, denoted by δS, is the set of all its boundary points.

Note that $$\delta S = \delta S^c.$$

We list below some useful properties which can be easily verified.

Properties: Let (X, d) be a metric space and S a subset of X. Then

(i) $S^0 = X - (\overline{X - S})$, $\overline{S} = X - (X - S)^0$, $\delta S = \overline{S} - S^0$;

(ii) S^0 is the largest open set contained in S;

(iii) $\overline{S}$ is the smallest closed set that contains S.

2.2 CONVERGENCE, COMPLETENESS

Let (X, d) be a metric space. Let $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is said to be **convergent** to x in X if for every $\varepsilon > 0$ there is an integer n_0 such that

$$d(x_n, x) < \varepsilon \text{ for } n \geq n_0.$$

In symbols, we write

$$\lim x_n = x.$$

Note that a sequence in a metric space can have at most one limit.

The following theorem characterizes the closure points of a set in terms of sequences.

Theorem 2.3. *Let (X, d) be a metric space and S a subset of X. Then a point $x \in X$ is a point of $\overline{S}$ iff there exists a sequence $\{x_n\}$ of S such that $\lim x_n = x$.*

In particular, if x is a limit point of S, then there exists a sequence of distinct points of S that converges to x.

Proof: Assume that $x \in \overline{S}$. For each n choose $x_n \in \overline{S}$ such that $d(x_n, x) < \dfrac{1}{n}$. It follows that $x_n \in S$ and limit $x_n = x$.

Conversely, assume that there exists a sequence $\{x_n\}$ of S such that $\lim x_n = x$.

Then for each r there exists n_0 such that $d(x_n, x) < r$ for $n > n_0$. Thus,

$$B_r(x) \cap S \neq \phi \text{ for each } r > 0. \text{ Hence } x \in \overline{S}.$$

Now for the particular case, assume that x is a limit point of S. Then choose $x_1 \in S$ such that $x_1 \neq x$ and $d(x, x_1) < 1$. Having chosen $x_1 \in (S - \{x\})$ such that $d(x, x_1) < 1$, choose $x_2 \in S$ such that $x_2 \neq x$ and $d(x, x_2) < 1/2$. In this way, having chosen $x_1, x_2, ..., x_n \in (S - \{x\})$, choose $x_{n+1} \in S - \{x\}$ such that

$$d(x_{n+1}, x) < \min\left(\frac{1}{n+1}, d(x_n, x)\right).$$

Then $\{x_n\}$ is a sequence of S where $x_n \neq x_m$ $(n \neq m)$ and $\lim x_n = x$.

Dense Set: Let (X, d) be a metric space and S a subset of X. The set S is called dense in X if $\overline{S} = X$.

It follows from Theorem 2.3 that a set S is dense in X iff for every $x \in X$ there exists a sequence $\{x_n\}$ of S such that $\lim x_n = x$.

Example: The set Q of rational numbers is dense in R.

Cauchy Sequence: Let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is a Cauchy sequence if for every $\varepsilon > 0$ there is an integer n_0 such that

$$d(x_m, x_n) < \varepsilon \text{ for } m, n \geq n_0.$$

Note that a convergent sequence in X is a Cauchy sequence in X but the converse is not true. As an example, consider $X = (0, \infty)$ with $d(x, y) = |x - y|$, and define the sequence by $x_n = 1/n$. Then the sequence $\{x_n\}$ is a Cauchy sequence in X that does not converge in X.

A metric space (X, d) is said to be **complete** if every Cauchy sequence in X converges in X. The simple examples of complete metric spaces are:

(*a*) R or C with the usual metric $d(x, y) = |x - y|$;

(*b*) R^n with the metric

$$d(x, y) = |x - y| = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}.$$

Observe that the set Q of rational numbers is not complete.

Theorem 2.4. *Let (X, d) be a complete metric space and let S a subset of X. Then S is closed iff S is a complete metric space with metric d.*

Proof: Assume that S is closed. Let $\{y_n\}$ be a Cauchy sequence in S. Then $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, $\{y_n\}$ converges in X to a point x in X. But since S is closed, $x \in S$. This shows that (S, d) is a complete metric space.

Conversely, assume that (S, d) is a complete metric space. If there is a sequence $\{y_n\}$ of S such that line $y_n = x$ in X, then $\{y_n\}$ is a Cauchy sequence in X. It is clear that $\{y_n\}$ is a Cauchy sequence in S, and since S is complete, it must converge to a unique point in S. This point must be x. Thus, $x \in S$, and S is therefore closed.

Diameter: Let (X, d) be a metric space and S a subset of X. The diameter of S, denoted by **diam** S, is defined as

$$\text{diam } S = \sup \{d(x, y) : x, y \in S\}.$$

The set S is called **bounded** if diam $S < +\infty$.

Theorem 2.5 (Cantor). *Let (X, d) be a complete metric space and let $\{F_n\}$ be a sequence of closed, non-empty subsets of X such that $F_n \supset F_{n+1}$ for each n. Further, let*

$$\lim (\text{diam } F_n) = 0$$

Then $\bigcap_{n=1}^{\infty} F_n$ is a single point.

Proof: Let $x_1, x_2 \in \bigcap_{n=1}^{\infty} F_n$. Then $x_1, x_2 \in F_n$ for each n. Hence

$$d(x_1, x_2) \leq \text{diam } (F_n) \text{ for each } n.$$

Thus, $d(x_1, x_2) = 0$ implies $x_1 = x_2$.

This shows that $\bigcap_{n=1}^{\infty} F_n$ cannot contain more than one point.

It remains to prove that $\bigcap_{n=1}^{\infty} F_n$ is non-empty. For each n, choose $x_n \in F_n$. Since lim (diam F_n) = 0, then for $\varepsilon > 0$, there is n_0 such that diam $(F_{n_0}) < \varepsilon$. Now if $m, n \geq n_0$, then $F_m \subset F_{n_0}$, $F_n \subset F_{n_0}$, and this implies that x_m and x_n are in F_{n_0}. Therefore, $d(x_m, x_n) < \varepsilon$ for $m, n \geq n_0$ which shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, it follows that there is a point $x \in X$ such that $\lim x_n = x$.

The proof will be complete if we show that $x \in \bigcap_{n=1}^{\infty} F_n$. Since $x_m \in F_n$ for $m \geq n$, it follows that $x \in \overline{F_n}$ for each n. But by assumption, each F_n is closed, hence $\overline{F_n} = F_n$. Thus, $x \in F_n$ for each n, and therefore,

$$x \in \bigcap_{n=1}^{\infty} F_n .$$

This important theorem is due to **G. Cantor.** Before we conclude this section, we shall define **"nowhere dense"** set.

Let (X, d) be a metric space and S a subset of X. The set S is said to be nowhere dense if its closure has an **empty interior.** In other words, the set S is nowhere dense if $(\overline{S})^\circ = \phi$.

A classical example of nowhere dense subset of the real line is the **Cantor Set.** We shall not describe it and readers may consult Rudin [13], Principles of Mathematical Analysis, or Bartle, [3]. The Elements of Real Analysis.

2.3 CONTINUOUS FUNCTIONS

Let (X_1, d_1) and (X_2, d_2) be metric spaces. The function $f : X_1 \to X_2$ is said to be **continuous** at $a \in X_1$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_2(f(x), f(a)) < \varepsilon \text{ whenever } d_1(x, a) < \delta.$$

Note that δ depends on ε as well as on a.

The function $f : X_1 \to X_2$ is said to be continuous if it is continuous at each point of X_1.

The following theorem characterizes continuous functions in terms of open sets.

Theorem 2.6. *Let (X_1, d_1) and (X_2, d_2) be metric spaces and let $f : X_1 \to X_2$ be a function. Then f is continuous iff $f^{-1}(G)$ is open in X_1 whenever G is open in X_2.*

Proof: Let f be continuous. If $f^{-1}(G)$ is empty, then the proof is trivial. Assume that it is non-empty. Let $x \in f^{-1}(G)$. Then $f(x) \in G$. Since G is open, there exists some $r > 0$ such that $B_r(f(x)) \subset G$. Now by the continuity of f, there exists an open ball $B_\delta(x)$ such that $f(B_\delta(x)) \subset B_r(f(x))$, and since $B_r(f(x)) \subset G$, it follows that $B_\delta(x) \subset f^{-1}(G)$. Hence, $f^{-1}(G)$ is open. Conversely, assume that $f^{-1}(G)$ is an open subset of X_1 whenever G is an open subset of X_2. Let $\varepsilon > 0$. Consider the open set $G = B_\varepsilon(f(x))$. Since $x \in f^{-1}(G)$, there exists some $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(G)$. Thus $f(B_\delta(x)) \subset B_\varepsilon(f(x))$. Hence f is continuous at x. This proves the theorem.

Uniform Continuity: Let (X_1, d_1) and (X_2, d_2) be metric spaces and let $f : X_1 \to X_2$ be a function. We say that f is **uniformly continuous** if for every $\varepsilon > 0$ there exists $\delta > 0$ (depending only on ε) such that

$$d_2(f(x_1)), (f(x_2)) < \varepsilon \quad \text{whenever } d_1(x_1, x_2) < \delta.$$

Note that every uniformly continuous function is continuous but the converse is not true. As an example, let $X_1 = (0, 1)$ and $X_2 = R$ both with $d(x, y) = |x - y|$. Then $f(x) = 1/x$ is continuous but not uniformly continuous.

2.4 COMPACTNESS

Let (X, d) be a metric space. A **collection** $\{G_i\}$ of open subsets of X is said to be an **open cover** of X if $X \subset \underset{i \in I}{\cup} G_i$. Let S be a subset of X. The set S is said to be **compact** if every open cover of S has **finite subcover.** In other words, there is a finite number of sets $G_1, G_2, ..., G_n$ in the collection $\{G_i\}$ such that

$$S \subset G_1 \cup G_2 \cup \ldots \ldots \cup G_n.$$

Note that the empty set and all finite sets are compact.

The set $S = \{z \in C : |z| < 1\}$ is not compact.

A metric space (X, d) is said to be **sequentially compact** if every sequence in it has a convergent **sub-sequence.**

The **Heine-Borel theorem** states that every closed and bounded set in R^n is compact. We prove here the converse result.

Theorem 2.7. *Let E be a subset of* R^n. *Then the following statements are equivalent.*

(*i*) *E is compact.*

(*ii*) *E is closed and bounded.*

Proof: As noted above, $(ii) \Rightarrow (i)$. If we prove that $(i) \Rightarrow (ii)$, this will establish the equivalence of the statements.

Assume (i) holds. We shall prove first that E is bounded. For each $k \in I$, let B_k be the open ball defined by

$$B_k = \{x \in R^n : |x| < k\}.$$

Clearly, $E \subset \underset{k \in I}{\cup} B_k$. Since E is compact, $E \subset \overset{m}{\underset{i=1}{\cup}} B_i$. This shows that there exists $m \in I$ such that $E \subset B_m$, and hence E is bounded.

Next we prove that E is closed. Suppose E is not closed. Then there is a limit point y of E such that $y \notin E$. Now for $x \in E$ and for $r_x = |x - y|/2$, construct the open ball $B_{r_x}(x)$, with centre at x and radius r_x. The collection of open balls $\{B_{r_x}(x) : x \in E\}$ is an open covering of E. By compactness of E, there exists a finite subcover, say $B_{r_1}(x_1)$, $B_{r_2}(x_2)$, ..., $B_{r_n}(x_n)$ such that

$$E \subset \bigcup_{i=1}^{n} B_{r_i}(x_i).$$

Let $r = \min\{r_1, r_2, ..., r_n\}$. Then it is easy to show that the open ball $B_r(y)$ has no points in common with any of the collection $B_{r_k}(x_k)$. In fact, if $x \in B_r(y)$, then

$$|x - y| < r \le r_k$$

and
$$\begin{aligned} |x - x_k| &= |y - x_k + x - y| \ge |y - x_k| - |x - y| \\ &= 2r_k - |x - y| > r_k, \end{aligned}$$

so that $x \notin B_{r_k}(x_k)$. Therefore, $B_r(y) \cap E$ is empty. This contradicts that y is a limit point of E. Henec E is closed.

Totally Bounded Set: Let (X, d) be a metric space and E a subset of X. The set E is said to be **totally bounded** if for every $r > 0$ there exists a finite number of points $x_1, x_2, ..., x_n$ such that

$$E \subset \bigcup_{i=1}^{n} B_r(x_i).$$

Note that a compact metric space is totally bounded but the converse is not necessarily true. As an example, take $E = (0, 1)$, $X = R$, with $d(x, y) = |x - y|$.

Theorem 2.8. *Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let $f : X_1 \to X_2$ be a continuous function, and let E be a compact subset of X_1. Then f(E) is a compact subset of X_2.*

Proof: Let $f(E) \subset \bigcup_{i \in I} G_i$ be an open cover. Then $E \subset \bigcup_{i \in I} f^{-1}(G_i)$. Since f is continuous, it follows from Theorem 2.6 that each of the sets $f^{-1}(G_i)$ is open. Since E is compact, there exist indices $i_1, i_2, ..., i_n$ such that

$$E \subset f^{-1}(G_{i_1}) \cup f^{-1}(G_{i_2}) \cup ... \cup f^{-1}(G_{i_n}).$$

Thus
$$\begin{aligned} f(E) &\subset f\left(\bigcup_{k=1}^{n} f^{-1}(G_{i_k})\right) \\ &= \bigcup_{k=1}^{n} f(f^{-1}(G_{i_k})) = \bigcup_{k=1}^{n} G_{i_k}, \end{aligned}$$

which shows that $f(E)$ is compact.

Theorem 2.9. *Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let $f : X_1 \to X_2$ be a continuous function, and let X_1 be compact. Then f is uniformly continuous.*

Proof: Let $\varepsilon < 0$. Since f is continuous, there exist $\delta_x > 0$ such that

(1) $d_2(f(x), f(y)) < \varepsilon/2$ whenever $d_1(x, y) < \delta_x$.

Let G_x be the set defined by

$$G_x = \left\{ y \in X_1 : d_1\,(x, y) < \frac{1}{2}\delta_x \right\}$$

Let $X_1 \subset \underset{i \in I}{\cup} (G_{x_i})$ be an open cover. Since X_1 is compact, there exist finite number of points $x_1, x_2, \ldots, x_n$ in X_1 such that

$$X_1 \subset \overset{n}{\underset{k=1}{\cup}} G_{x_k}$$

Let $\delta = \min\,\{\delta_{x_1}/2, \delta_{x_2}/2, \ldots, \delta_{x_n}/2\}$.

Then $\delta > 0$. Note that δ does not depend on x.

Now if x and y are any two points in X_1 such that $d_1(x, y) < \delta$, then $x \in G_{x_m}$ for at least one G_{x_m}. This shows that $d_1(x, x_m) < \frac{1}{2}\delta_{x_m}$. Also,

$$\begin{aligned} d_1\,(x_m, y) &\le d_1\,(x_m, x) + d_1(x, y) \\ &< \frac{1}{2}\delta_{x_m} + \delta \le \delta_{x_m}. \end{aligned}$$

It now follows from (1) that

(2) $\quad d_2\,(f(y), f(x_m)) < \varepsilon/2$ whenever $d_1\,(x_m, y) < \delta_{x_m}$.

Thus $\quad d_2\,(f(x), f(y)) \le d_2\,(f(x), f(x_m)) + d_2\,(f(x_m), f(y))$
$< \varepsilon/2 + \varepsilon/2 = \varepsilon,$

so that f is uniformly continuous.

This completes the proof of the theorem.

Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let $f : X_1 \to X_2$ be a function, and let E be a subset of X_1. We call f an **open mapping** if $f(E)$ is open whenever E is open. Similarly, we call f a **closed mapping** if $f(E)$ is closed whenever E is closed.

Theorem 2.10. *Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let $f : X_1 \to X_2$ be a continuous function, and let X_1 be compact. Then f is a closed mapping.*

Proof: Let E be a closed subset of X_1. We need to prove first that E is a compact subset of X_1.

Let $\{G_i : i \in I\}$ be an open cover of E. Then $X_1 = E^c \cup \left[\underset{i \in I}{\cup} G_i\right]$ is an open cover of X_1. By the compactness of X_1, there exist indices $i_1, i_2, \ldots, i_n$ such that

$$X_1 = E^c \cup (G_{i_1} \cup G_{i_2} \cup \ldots \cup G_{i_n}).$$

This shows that $E \subset G_{i_1} \cup G_{i_2} \cup \ldots \cup G_{i_n}$,

and so, E is compact.

By Theorem 2.8, the set $f(E)$ is a compact subset of X_2. Hence, $f(E)$ is closed, and therefore, f is a closed mapping. This completes the proof of the theorem.

We say that two metric spaces (X_1, d_1) and (X_2, d_2) are **homeomorphic** if there exist a one-to-one and onto function $f : X_1 \to X_2$ such that f and f^{-1} are both continuous.

Observe that in Theorem 2.10 if we assume further that f is one-to-one and onto, then f is a **homeomorphism.**

2.5 CONNECTEDNESS

Definition: Let (X, d) be a metric space and S a subset of X. The set S is said to be **disconnected** if there exist two disjoint non-empty open subsets G and H of X such that G intersects S and H intersects S, and $S \subset G \cup H$. In this case the pair G, H is said to form a disconnection of S.

A set S in a metric space X is said to be **connected** if it is not disconnected.

Examples: (*a*) The set I of all positive integers is disconnected in R.

Let $G = \left\{x \in R : x < \dfrac{3}{2}\right\}$ and $H = \left\{x \in R : x > \dfrac{3}{2}\right\}$.

It is immediately verified that the pair G, H form a disconnected of I.

(*b*) The set E of all positive rational numbers is disconnected in R.

Let $G = \{x \in R : x < \sqrt{2}\}$ and $H = \{x \in R : x > \sqrt{2}\}$.

The pair G, H form a disconnection of E.

(*c*) Consider the sets $G = \{x \in R : -1 < x \le u\}$, $H = \{x \in R : u < x < 2\}$ where $0 < u < 1$. Then G and H split the interval $[0, 1]$ into two disjoint non-empty subsets whose union is $[0, 1]$. But since G is not open it does not follows that $[0, 1]$ is disconnected. We shall prove that $[0, 1]$ is a connected set in R.

Note that in order to show that a set is connected we need to show that no disconnection can exist. The following theorem establishes a simple characterization of connected sets in R.

Theorem 2.11. *A subset S of the real line R is connected if and only if S has the following property.*

If $a \in S$, $b \in S$, *and* $a < c < b$, *then* $c \in S$.

Proof: The proof is by contradiction. Assume that $a \in S$, $b \in S$, $a < c < b$, but $c \notin S$. Let G be the set of all $x < c$ and H be the set of all $y > c$, then by the definition of connectedness it follows that S is not connected. Let $a \in S$, $b \in S$ and $a < b$, and suppose that S is disconnected. Then there exist disjoint open sets G, H in R such that $a \in G$, $b \in H$, $S \subset G \cup H$.

Let $c = \sup (G \cap [a, b])$

Since $b \in H$ and H is open, therefore $c < b$. If $c \in G$, then $c < b$; since G is open, there are points in $G \cap [a, b]$ which exceed c, contrary to its definition. Hence $c \notin G$.

Since $a \in G$ and G is open, therefore $a < c$. If $c \in H$, then since H is open there is a point $c_1 < c$ such that the interval $[c_1, c]$ is contained in $H \cap [a, b]$. Consequently,

$$[c_1, c] \cap G = \phi.$$

This also contradicts the definition of c as sup $(G \cap [a, b])$. Hence $c \notin H$. Since $S \subset G \cup H$, it follows that $c \notin S$, which proves the theorem.

Corollary: *A subset S of R is connected if and only if S is an interval. In particular, R is connected.*

There is no such simple characterization of connected sets in the complex plane.

Connectedness in *R^n*

Theorem 2.12. *The space R^n is connected.*

Proof: Assume that R^n is disconnected. Then there exist two disjoint non-empty open sets A, B such that $R^n \subset A \cup B$. Let $a \in A$ and $b \in B$. Define the line segment E joining a and b by

$$E = \{a + t(b - a) : t \in [0, 1]\}.$$

Let $\quad G = \{t \in R : a + t\,(b - a) \in A\}$

and $\quad H = \{t \in R : a + t\,(b - a) \in B\}$.

It can be easily verified that G and H are disjoint non-empty open subsets of R and the pair G, H form a disconnection for [0, 1]. This contradicts that [0, 1] is connected.

Corollary: *The only subsets of R^n which are both open and closed are the empty set ϕ and the set R^n itself.*

The next theorem characterizes connectedness in R^n.

Definition: Let x and y be two points in R^n. **A polygonal curve** in R^n from a point x to a point y is an ordered set of a finite number of line segments $\{L_1, L_2, ..., L_n\}$ such that (*a*) x is the beginning point of L_1, (*b*) y is the end point of L_n, and (*c*) the end point of L_i is the beginning point of L_{i+1}.

This definition is illustrated in Fig. 2.II.

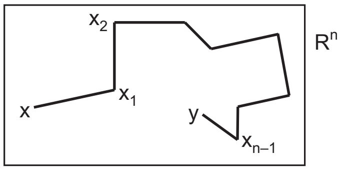


Fig. 2.II

Theorem 2.13. *Let E be an open set in R^n. Then E is connected iff any pair of points x, y in E can be joined by a polygonal curve which lies entirely in E.*

Proof: Assume that E is disconnected. Then there exist two open sets A and B which form a disconnection for E. Let $x \in A \cap E$ and $y \in B \cap E$. Let x and y be joined by a polygonal curve $(L_1, L_2, ..., L_n)$ which lies entirely in E.

Let $i \in I$ be the smallest number such that the end point x_{i-1} of L_i belongs to $A \cap E$ and the end point x_i belongs to $B \cap E$. Define

$$G = \{t \in R : x_{k-1} + t(x_k - x_{k-1}) \in A \cap E\}$$

and $$H = \{t \in R : x_{k-1} + t(x_k - x_{k-1}) \in B \cap E\}.$$

Then it can be easily verified that G and H are disjoint non-empty open subsets of R and the pair G, H form a disconnection for [0, 1]. This contradicts the fact that [0, 1] is connected. Thus, if E is disconnected, then any two points in E cannot be joined by a polygonal curve lying entirely in E.

Conversely, assume that E is connected open set in R^n. Let $x \in E$. Let S be the set of all those points of E which can be joined to x by a polygonal curve lying entirely in E. Let T be the set of all those points of E which cannot be joined to x by a polygonal curve lying entirely in E. Clearly, $S \cap T = \phi$. The set S is non-empty since it contains the point x. We now prove that S is open in R^n. Let $y \in S$. Since E is open there exist some positive number r such that $|z - y| < r$, and so, $z \in E$. It now follows from the definition of S that if a segment from y to z is added to a polygonal curve from y to x, then $z \in S$. This shows that S is open in R^n. Similarly, we can prove that T is open in R^n.

If T is not empty, then the pair S, T form a disconnection of E and this contradicts that E is connected. Therefore, $T = \phi$ and every point of E can be joined to x by a polygonal curve which lies entirely in E. This completes the proof of the theorem.

Examples: (*a*) Consider the set

$$G = \{(x, y) \in R^2 : 0 < y \le x^2, x \ne 0\} \cup \{(0, 0)\}.$$

The set G is connected in R^2, but it is not true that any two points in G can be joined by a polygonal curve lying entirely in G.

(*b*) Consider the set

$$H = \left\{(x, y) \in R^2 : y = \sin\left(\frac{1}{x}\right), x \ne 0\right\} \cup \{(0, y) : -1 \le y \le 1\}.$$

The set H is connected in R^2, but it is not true that every pair of points in H can be joined by a polygonal curve lying entirely in H.

Theorem 2.14. *Let (X_1, d_1) and (X_2, d_2) be metric spaces: Let $f : X_1 \to X_2$ be a continuous function. If X_1 is connected, then $f(x_1)$ is a connected subset of X_2.*

Proof: Assume that $f(x_1)$ is not connected. Then there exist disjoint non-empty open sets A and B in X_2, both of which intersect $f(X_1)$ and

$$f(X_1) \subset A \cup B.$$

Since f is continuous, it follows from Theorem 2.6 that $f^{-1}(A)$ and $f^{-1}(B)$ are open in X_1. It is immediately verified that $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint and non-empty and

$$f^{-1}(A) \cup f^{-1}(B) = X_1.$$

This shows that X_1 is disconnected, which is a contradiction.

Note that if $E \subset X_2$ is connected and f is continuous, then it is not necessarily true that $f^{-1}(E)$ is connected in X_1. But if we take disjoint segments in the complex plane C and project them on the real axis, then the projection is connected. The following figure illustrates this fact.

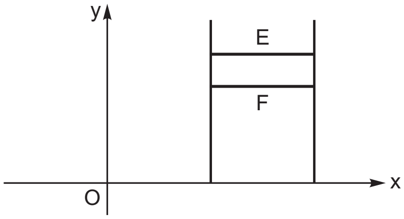


Fig. 2.III

Convex Set: Let S be a subset of R^n. The set S is said to be **convex** if $\lambda x + (1 - \lambda)y \in S$ whenever $x \in S$, $y \in S$, and $0 < \lambda < 1$.

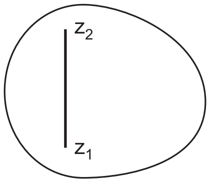
Example: Open and closed balls in R^n are convex sets.

Region: Let S be a subset of the complex plane C. If S is open and connected, then S is called a region.

Examples: (*a*) An open convex set is a region.

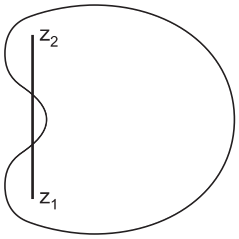
(*b*) An open disc, an **open ellipse,** an **open triangle** and an **open square** are **convex regions.**

(*c*) We say that an open set $S \subset C$ is **starlike** if there exists a point $z_0 \in S$ such that for any $z \in S$ the **segment** $[z_0, z]$ belongs to S. The point z_0 is referred to as the **centre of the star.**



Convex region

Fig. 2.IV

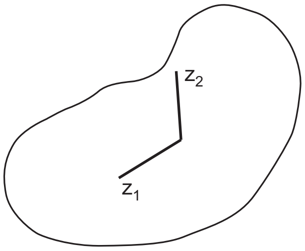


Non-convex region

Fig. 2.V

It is immediately verified that an **open starlike set** is connected, and so, is a region.

We illustrate these examples in the following figure.



Starlike region

Fig. 2.VI

EXERCISES

1. Let (X, d) be a metric space. Prove that every open ball is open set and every closed ball is a closed set.
2. Prove that a set $E \subset X$ is open iff $X - E$ is closed.
3. Let (X, d) be a metric space and $Y \subset X$. Suppose that $E \subset X$ is open. Prove that $E \cap Y$ is open in (Y, d). Conversely, show that if $E_1 \subset Y$ is open in (Y, d), there is an open set $E \subset X$ such that $E_1 = E \cap Y$.
4. Let (X, d) be a metric space. Prove that $\left(X, \dfrac{d}{1+d}\right)$ is also a metric space.
5. Let (X, d) be a metric space. Prove that any two distinct points of X can be separated by open balls in the following sense: if x and y are distinct points in X, then there exists a disjoint pair of open balls each of which is centred on one of the points.
6. Find the interior of each of the following subsets of the real line; the set of all integers; the set of all rationals; the set of all irrationals $(0, 1)$; $[0, 1]$; $[0, 1) \cup \{1, 2\}$. Find also the interior of each of the following subsets of the complex plane: $\{z : |z| < 1\}$; $\{z : |z| \leq |1\}$; $\{z : I(z) = 0\}$; $\{z : R(z) \text{ is rational}\}$.
7. Show that a subset of a metric space is bounded iff it is non-empty and is contained is some closed ball.
8. Find the boundary of each of the following subsets of the real line: the integers; the rationals; $[0, 1]$; $(0, 1)$. Find also the boundary of each of the following subsets of the complex plane: $\{z : |z| < 1\}$; $\{z : |z| \leq |1\}$; $\{z : I(z) > 0\}$.
9. Prove that a Cauchy sequence is convergent iff it has a convergent subsequence.
10. Give three examples of metric spaces which are not complete.
11. Let X and Y be metric spaces and let f be a mapping of X into Y. If f is a constant mapping, prove that f is continuous. Use this to show that a continuous mapping need not have the property that the image of every open set is open.
12. Let X and Y be metric spaces and f a mapping of X into Y. Show that f is continuous iff $f^{-1}(F)$ is closed in X whenever F is closed in Y iff $f(\overline{B}) \subseteq \overline{f(B)}$ for every subset B of X.

13. Examine the uniform continuity of the following functions on the open unit interval (0, 1)

(*i*) $\dfrac{1}{1-x}$ (*ii*) $\sin x$

(*iii*) $\sin\left(\dfrac{1}{x}\right)$ (*iv*) $x^{1/2}$

(*v*) x^3

Examine also which are uniformly continuous in the open interval $(0, +\infty)$.

14. Let (X, d_1) and (Y, d_2) be metric spaces. Suppose that $f : X \to Y$ is uniformly continuous. Show that if $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is a Cauchy sequence in Y. Does the result hold good if we only assume that f is continuous.

15. Suppose that (Y, d_2) is a complete metric space and suppose that $f : (Z, d_3) \to (Y, d_2)$ is uniformly continuous, where Z is dense in (X, d_1). By using exercise 14 show that there is uniformly continuous function $g : X \to Y$ with $g(x) = f(x)$ for every $x \in Z$.

16. Let E be a compact subset of the metric space (X, d) and let $x \in X$. Define the distance dist. $(x, E) = \inf\{d(x, y) : y \in E\}$. Show that if $x \notin E$, then dist. $(x, E) > 0$.

17. Let E and F be two compact subsets of the metric space (X, d). Define the distance dist. (E, F) by dist. $(E, F) = \inf\{d(x, y) : x \in E, y \in F\}$. Show that if $E \cap F = \phi$, then dist. $(E, F) > 0$.

18. Prove that the union of a finite number of compact sets is compact.

19. Let (X, d) be a metric space and let $E \subset X$. Prove that if E is connected, then its closure $\overline{E}$ is also connected.

20. Let X be a **vector space.** A **real valued function** $\|\cdot\|$ defined on X is called a **norm** if it satisfies the following properties:

(*i*) $\|x\| \geq 0$ for each $x \in X$, and $\|x\| = 0$ iff $x = 0$;

(*ii*) $\|\alpha x\| = |\alpha|\,\|x\|$ for all $x \in X$ and $\alpha \in R$;

(*iii*) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Property (*iii*) is called the **triangle inequality.**

Note that the absolute value function yields a norm for R or C.

A vector space together with a norm for X is called a **normed space.**

On a normed space X a metric is defined by

$$d(x, y) = \|x - y\|.$$

(*a*) Verify that $d(x, y)$ is indeed a metric on X.

(*b*) Show that $|\,\|x\| - \|y\|\,| \leq \|x - y\|$.

(*c*) Show that the vector space R^n with the norm

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

for each $x = (x_1, x_2, ..., x_n) \in R^n$ is a normed space.

(*d*) Let $\mathcal{B}(X)$ be the vector space of all **bounded real valued functions** defined on a non-empty set X. Define a norm on $\mathcal{B}(X)$ by

$$\| f \|_\infty = \sup \{ | f(x) | : x \in X \}$$

for each $f \in \mathcal{B}(X)$. This norm is referred to as the **sup norm**.

Prove that the vector space $\mathcal{B}(X)$ with the sup norm is a normed space.

(*e*) Let X be a vector space. We say that two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on X are equivalent if there exist constants $K > 0$ and $M > 0$ such that

$$K \| x \|_1 \leq \| x \|_2 \leq M \| x \|_1$$

for every $x \in X$.

Let $$v = (x, y) \in R^2$$

Define the following norms on R^2 by

(*i*) $\| v \|_1 = | x | + | y |$;

(*ii*) $\| v \|_2 = (x^2 + y^2)^{1/2}$;

(*iii*) $\| v \|_\infty = \max \{ | x |, | y | \}$;

(*iv*) $\| v \| = \left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \right)^{1/2}$,

where a and b are two fixed positive numbers.

Prove that the above norms are all equivalent.

Illustrate geometrically the closed unit ball for each norm described above.

3

ELEMENTARY PROPERTIES OF ANALYTIC FUNCTIONS

3.1 LIMITS AND CONTINUITY

Let Ω be an open subset of the complex plane C. Let f be a function on Ω. Let A be a complex constant. We say that

$$\lim_{z \to z_0} f(z) = A$$

if the following condition is satisfied.

Given $\varepsilon > 0$ there exists a number $\delta > 0$ such that if $z \in \Omega$ and $|z - z_0| < \delta$, then

$$|f(z) - A| < \varepsilon.$$

Note that this definition of a limit implies that z may approach z_0 from any direction in the complex plane C. For example, let us take

$$f(z) = \frac{(x+y)^2}{x^2 + y^2}.$$

It can be easily seen that

$$\lim_{x \to 0}\left[\lim_{y \to 0} f(z)\right] = 1 = \lim_{y \to 0}\left[\lim_{x \to 0} f(z)\right].$$

But along the path $y = mx$, we have

$$\lim_{z \to 0} f(z) = \frac{(1+m)^2}{1+m^2}.$$

The limiting value here depends on m and hence

$$\lim_{z \to 0} f(z) \text{ does not exist.}$$

Let f be a function on Ω. Let $z_0 \in \Omega$. We say that f is **continuous** at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

If f is continuous at every point of Ω, we say that f is continuous on Ω.

Examples: (*i*) The function $f(z) = z$, $z \in C$ is continuous on C.

(*ii*) The functions Re z and Im z are continuous on C.

(*iii*) The function $f(z) = \dfrac{1}{z}$ is coutinuous on $C - \{0\}$.

(*iv*) The function $f(z) = \begin{cases} |z|, z \neq 0 \\ 1, z = 0 \end{cases}$ is not continuous at the origin.

The sum and product of two continuous functions are continuous; the quotient f/g is continuous at z_0 provided $g(z_0) \neq 0$.

If $r > 0$ and z_1 is a complex number. We denote

$$D(z_1, r) = \{z : |z - z_1| < r\}.$$

$D(z_1, r)$ is called the **open circular disc** † with centre at z_1 and radius r. The closure of $D(z_1, r)$ is

$$\overline{D}(z_1, r) = \{z : |z - z_1| \leq r\}.$$

We also denote

$$D'(z_1, r) = \{z : 0 < |z - z_1| < r\}$$

which is called the **punctured disc** with centre at z_1 and radius r.

3.2 COMPLEX DIFFERENTIABILITY

Let Ω be an open set in the complex plane C. Suppose f is a function on Ω. The function f is said to be **differentiable** at a point z_0 of Ω if

$$(1)\ \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The derivative is denoted by $f'(z_0)$. Note that the limit is independent of the path along which $z \to z_0$ in the complex plane.

Writing $z - z_0 = h$ in the above definition, we get the equivalent form

$$(1')\ \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0).$$

It is important to note that h is a complex number. Replacing z_0 by z in $(1')$ we have

$$(1'')\ \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = f'(z).$$

Denote h by Δz. Then $(1'')$ takes the form

$$(1''')\ \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = f'(z).$$

Suppose $w = f(z)$. We sometimes define

$$\Delta w = f(z + \Delta z) - f(z)$$

---

† The open circular disc is also called a neighbourhood of the point z_1.

and write the derivative as

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}.$$

By the definition of a limit, (1) means that $f'(z_0)$ exists if to every $\varepsilon > 0$ there corresponds a $\delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

for all $z \in D'(z_0, \delta)$.

Analytic Functions: A function f is said to be **analytic** at a point z_0, if it is differentiable throughout some ε – **neighbourhood** of z_0. A function f is analytic in a region if it is analytic at every point of the region.

Observe that a function which is differentiable at a point, need not necessarily be analytic at that point. Example (*v*) below illustrates this fact.

Examples (*i*) If n is a positive integer, then $f(z) = z^n$ is differentiable in the entire complex plane.

(*ii*) $f(z) = \text{Re } z$ and $f(z) = \text{Im } z$ are not differentiable.

(*iii*) $f(z) = \dfrac{1}{2}$ is differentiable in $C - \{0\}$.

(*iv*) $f(z) = \bar{z}$ is not differentiable.

(*v*) $f(z) = |z|^2$ is differentiable only at $z = 0$.

Theorem 1. *If* $f : \Omega \to C$ *is differentiable at a point* z_0 *in* Ω*, then* f *is continuous at* z_0.

Proof: $$\lim_{z \to z_0} |f(z) - f(z_0)| = \left[\lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} \right] \left[\lim_{z \to z_0} |z - z_0| \right]$$

$$= f'(z_0) \cdot 0 = 0.$$

Hence f is continuous at z_0.

Let f and g be functions defined on the open set Ω. Suppose that f and g are differentiable at z. Then

(*i*) the sum $f + g$ is differentiable at z and

$$(f + g)'(z) = f'(z) + g'(z);$$

(*ii*) the product fg is differentiable at z, and

$$(fg)'(z) = f'(z)\, g(z) + f(z)\, g'(z);$$

(*iii*) the quotient f/g is differentiable at z provided $g(z) \neq 0$, and

$$(f/g)'(z) = \frac{g(z)\, f'(z) - f(z)\, g'(z)}{(g(z))^2}$$

The proof is left to the reader which is well known in real analysis.

Chain Rule. Let f and g be differentiable on Ω_1 and Ω_2 respectively and let $f(\Omega_1) \subset \Omega_2$. Then $g \circ f$ is differentiable on Ω_1 and

$$(g \circ f)'(z) = g'(f(z))\, f'(z).$$

Proof: In order to prove the chain rule we need the following relations. Let f be differentiable on Ω, $z \in \Omega$, and $h \neq 0$. Consider

$$\eta(z, h) = \frac{f(z+h) - f(z)}{h} - f'(z).$$

Then $$f(z+h) - f(z) = h f'(z) + h\eta(z, h)$$

and $$\lim_{h \to 0} \eta(z, h) = 0.$$

Conversely, suppose that

$$f(z+h) = f(z) + h f_1(z) + h\,\zeta(z, h)$$

where $\lim_{h \to 0} \zeta(z, h) = 0$.

Then f is differentiable at z with

$$f_1(z) = f'(z).$$

We apply the above relations to a proof of the chain rule.

Let $D(z, r) \subset \Omega_1$ and $|h| < r$.

Then, since f is differentiable, we have

$$k = f(z+h) - f(z),$$

where $$k = hf'(z) + h\,\eta(z, h)$$

and $$\lim_{h \to 0} \eta(z, h) = 0.$$

Hence $g(f(z+h)) = g(f(z) + k)$

$$= g(f(z)) + kg'(f(z)) + k\zeta(f(z), k),$$

where $\lim_{k \to 0} \zeta(f(z), k) = 0$.

Substituting for k its expression in terms of h, we have

$$\eta(z, h)\, g'(f(z)) + (f'(z) + \eta(z, h))\, \zeta(f(z), k)$$
$$= \xi(z, h) \text{ (say)}.$$

Then

$$g(f(z+h)) = g(f(z)) + hg'(f(z))\, f'(z) + h\,\xi(z, h).$$

It can be easily seen that

$$\lim_{h \to 0} \xi(z, h) = 0.$$

Therefore

$$\lim_{h \to 0} \frac{g(f(z+h)) - g(f(z))}{h} = g'(f(z))\, f'(z)$$

i.e. $$(g \circ f)'(z) = g'(f(z))\, f'(z).$$

3.3 THE CAUCHY-RIEMANN EQUATIONS

Any complex-valued function may be decomposed into its real part, $\operatorname{Re} f(z) = u(x, y)$ and its imaginary part, $\operatorname{Im} f(z) = v(x, y)$ such that

$$f(z) = u(x, y) + iv(x, y).$$

Here u and v are real valued functions of the two real variables x and y.

We shall now derive a simple basic criterion for analyticity of a complex valued function.

Let $f : \Omega \to C$ be analytic.

By definition,

$$(1)\ f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

(2)

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)\} - \{u(x, y) + iv(x, y)\}}{\Delta x + i\,\Delta y}$$

If $\Delta y = 0$, we obtain

$$f'(z) = \lim_{\Delta x \to 0} \left[\frac{\{u(x + \Delta x, y) + iv(x + \Delta x, y)\} - \{u(x, y) + iv(x, y)\}}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i\,\frac{\{v(x + \Delta x, y) - v(x, y)\}}{\Delta x}\right]$$

$$(3) \qquad = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}.$$

On the other hand, if $\Delta x = 0$, we get from (2)

$$f'(z) = \lim_{\Delta y \to 0} \left[\frac{\{u(x, y + \Delta y) + iv(x, y + \Delta y)\} - \{u(x, y) + iv(x, y)\}}{i\Delta y}\right]$$

$$= \lim_{\Delta y \to 0} \left[\frac{\{u(x, y + \Delta y) - u(x, y)\}}{i\Delta y} + i\,\frac{\{v(x, y + \Delta y) - v(x, y)\}}{i\Delta y}\right]$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

$$(4) \qquad = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}.$$

Hence, from (3) and (4)

$$\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}.$$

Thus

$$(5) \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These basic relations are called **Cauchy-Riemann equations.**

Suppose that u and v have continuous second partial derivatives. Differentiating the $C-R$ equations again we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\, \partial y} \text{ and } \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y\, \partial x}\cdot$$

Hence,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{6}$$

Any function satisfying (6) is said to be **harmonic.** We shall discuss harmonic functions in Chapter 10.

Let Ω be a region in the plane and let u and v be functions defined on Ω with continuous partial derivatives. Suppose also that u and v satisfy the $C-R$ equations.

If $\quad f(z) = u(x, y) + iv(x, y),$

then f can be shown to be analytic in Ω.

To see this, let $z = x + iy \in \Omega$ and let $B_r(z) \subset \Omega$.

If $\Delta z = \Delta x + i\Delta y \in B_r(0)$, then

$$u(x + \Delta x, y + \Delta y) - u(x, y)$$
$$= [u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y)] + [u(x, y + \Delta y) - u(x, y)]. \tag{7}$$

By the mean value theorem for the derivative of a function of one variable, we have

$$\begin{cases} u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y) = u_x\,(x + \Delta x_1, y + \Delta y)\,\Delta x \\ u(x, y + \Delta y) - u(x, y) = u_y\,(x, y + \Delta y_1)\,\Delta y \end{cases} \tag{8}$$

where $|\Delta x_1| < |\Delta x|$ and $|\Delta y_1| < |\Delta y|$.

Denote

$$\phi(\Delta x, \Delta y) = [u(x + \Delta x, y + \Delta y) - u(x, y)] - [u_x(x, y)\,\Delta x + u_y(x, y)\,\Delta y].$$

Now from (8) we obtain

$$\frac{\phi(\Delta x, \Delta y)}{\Delta x + i\Delta y} = \frac{\Delta x}{\Delta x + i\Delta y}\,[u_x(x + \Delta x, y + \Delta y) - u_x(x, y)]$$
$$+ \frac{\Delta y}{\Delta x + i\Delta y}\,[u_y(x, y + \Delta y) - u_y(x, y)]$$

But $|\Delta x| \le |\Delta x + i\Delta y|$, $|\Delta y| \le |\Delta x + i\,\Delta y|$, $|\Delta x_1| < |\Delta x|$, $|\Delta y_1| < |\Delta y|$ and the fact that u_x and u_y are continuous gives that

$$\lim_{\Delta z \to 0} \frac{\phi(\Delta x, \Delta y)}{\Delta z} = 0. \tag{9}$$

Hence

$$u(x + \Delta x, y + \Delta y) - u(x, y) = u_x(x, y)\,\Delta x + u_y(x, y)\,\Delta y + \phi(\Delta x, \Delta y)$$

where ϕ satisfies (9).

Similarly

$$v(x+\Delta x, y+\Delta y) - v(x, y) = v_x(x, y)\,\Delta x + v_y(x, y)\,\Delta y + \psi(\Delta x, \Delta y)$$

where ψ satisfies

$$\lim_{\Delta z \to 0} \frac{\psi(\Delta x, \Delta y)}{\Delta z} = 0 \tag{10}$$

Since $u(x, y)$ and $v(x, y)$ satisfy the $C-R$ equations, it can be easily verified that

$$\frac{f(z+\Delta x + i\,\Delta y) - f(z)}{\Delta x + i\Delta y} = u_x(x, y) + iv_x(x, y) + \frac{\phi(\Delta x, \Delta y) + \psi(\Delta x, \Delta y)}{\Delta x + i\Delta y}.$$

By (9) and (10), we see that f is differentiable and

$$f'(z) = u_x(x, y) + iv_x(x, y).$$

Since u_x and v_x are continuous, f' is continuous and thus f is analytic.

We summarise these results in the following theorem.

Theorem 2. *Let u (x, y) and v (x, y) be real valued functions defined on a region* Ω *and suppose that u (x, y) and v (x, y) have continuous partial derivatives. Then*

$$f : \Omega \to C \text{ defined by}$$

$$f(z) = u(x, y) + iv(x, y)$$

is analytic iff u and v satisfy the C – R equations.

3.4 EXPONENTIAL FUNCTION

The **exponential function** e^z is of basic importance in complex analysis. In fact, it serves as a basis for defining all other elementary **transcendental functions.** In calculus we have seen that the real exponential function e^x (also written exp x) has the properties

$$\frac{d}{dx}(e^x) = e^x$$

and

$$e^{x_1 + x_2} = e^{x_1}\, e^{x_2}.$$

While defining e^z we will preserve as many of the familiar properties of the real exponential function e^x. We wish that

(a) e^2 shall be single-valued and analytic;

(b) $\frac{d}{dz}(e^z) = e^z$;

(c) $e^z = e^x$ when Im $z = 0$.

Let

(11) $e^z = u + iv$.

In order that e^z should satisfy condition (*b*), we have

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u + iv.$$

Hence

(12) $\dfrac{\partial u}{\partial x} = u.$

(13) $\dfrac{\partial v}{\partial x} = v.$

Now (12) will be satisfied if

(14) $u = e^x \phi(y)$

where $\phi(y)$ is any function of y. Furthermore, since e^z is to be analytic, u and v must satisfy the $C - R$ equations.

Hence

(15) $-\dfrac{\partial u}{\partial y} = v.$

Differentiating this with respect to y; we obtain

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial v}{\partial y}$$

i.e.
$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial u}{\partial x}.$$

Using (12), this becomes

$$\frac{\partial^2 u}{\partial y^2} = -u$$

which, on substituting $u = e^x \phi(y)$ from (14), reduces to

$$e^x \phi''(y) = -e^x \phi(y)$$

i.e.
$$\phi''(y) = -\phi(y).$$

This is a simple linear differential equation whose solution is

$$\phi(y) = A\cos y + B\sin y.$$

Hence, from (14),

$$u = e^x \phi(y) = e^x (A\cos y + B\sin y)$$

and from (15),
$$v = -\frac{\partial u}{\partial y} = -e^x(-A\sin y + B\cos y).$$

Therefore, from (11)

$$e^z = u + iv = e^x [(A\cos y + B\sin y) + i(A\sin y - B\cos y)]$$

This reduces to e^x when $y = 0$, as required by condition (*c*).

Hence $e^x = e^x(A - iB)$

which holds iff $A = 1$ and $B = 0$.

Thus we arrive at the definition that if there is a function of z satisfying the conditions (a), (b) and (c), then it must be

(16) $e^z = e^{x+iy} = e^x (\cos y + i \sin y)$.

Note that

$$|e^z| = e^x \text{ and } \arg e^z = y.$$

The possibility of writing any complex number in exponential form is now apparent.

Applying (16) with $x = 0$ and $y = \theta$, we have

(17) $e^{i\theta} = \cos\theta + i \sin\theta$.

Thus $$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

A function f is periodic with period c if $f(z + c) = f(z)$ for all $z \in C$. If c is a period of e^z then

$$e^z = e^{z+c} = e^z e^c \text{ implies that } e^c = 1.$$

Since $1 = |e^c| = e^{\text{Re}(c)}$, $\text{Re}(c) = 0$.

Thus $c = i\theta$ for some θ in R.

But $1 = e^c = e^{i\theta} = \cos\theta + i \sin\theta$ gives that the periods of e^z are the integral multiples of $2\pi i$. Thus, if we divide the plane into infinitely many horizontal strips by the lines $z = 2\pi ik$, k being any integer, the exponential function behaves the same in each of these strips. This property of periodicity is one which is not present in the real exponential function.

3.5 TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

On the basis of equation (17) we extend the definitions of $\cos z$ and $\sin z$

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz});$$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

From these definitions one can establish the familiar formulas:

$$\cos^2 z + \sin^2 z = 1.$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2.$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2.$$

$$\frac{d}{dz}(\cos z) = -\sin z.$$

$$\frac{d}{dz}(\sin z) = \cos z.$$

Expanding $\cos z$ in exponentials, we have

$$\cos z = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2}$$

$$= \frac{e^{-y}(\cos x + i \sin x) + e^{y}(\cos x - i \sin x)}{2}$$

$$= \cos x \, \frac{e^{y} + e^{-y}}{2} - i \sin x \, \frac{e^{y} - e^{-y}}{2}.$$

Using the usual definitions of the **hyperbolic functions** of real variables, we get

(18) $\cos z = \cos (x + iy) = \cos x \cosh y - i \sin x \sinh y$.

Similarly

(19) $\sin z = \sin (x + iy) = \sin x \cosh y + i \cos x \sinh y$.

In particular, taking $x = 0$ in (18) and (19), we obtain

(20) $\cos iy = \cosh y$;

(21) $\sin iy = i \sinh y$.

The remaining **trigonometric functions** of z are defined in terms of $\cos z$ and $\sin z$ by means of the usual identities.

We define the hyperbolic functions of z by

(22) $\cosh z = \dfrac{e^{z} + e^{-z}}{2}$;

(23) $\sinh z = \dfrac{e^{z} - e^{-z}}{2}$.

By expanding the exponentials and simplifying we get

(24) $\cosh z = \cosh x \cos y + i \sinh x \sin y$;

(25) $\sinh z = \sinh x \cos y + i \cosh x \sin y$.

Setting $x = 0$, we obtain

(26) $\cosh iy = \cos y$;

(27) $\sinh iy = i \sin y$.

3.6 LOGARITHM

Let Ω be a region of C. We define a branch of the logarithm in Ω to be a continuous function $f : \Omega \to C$ such that

(28) $\exp (f(z)) = z$ for all $z \in \Omega$.

Setting $z = re^{i\theta}$ and $w = f(z) = u + iv$ in (28), we have

$$e^{u} = r, \text{ i.e. } u = \ln r \text{ and } v = \theta.$$

Thus $\quad w = u + iv = \ln r + i\theta.$

(29) $w = \ln |z| + i \arg z$.

If we let θ_1 be the principal argument of z, (29) can be written

(30) $\ln z = \ln |z| + i(\theta_1 + 2n\pi)$, $(n = 0, \pm 1, \pm 2, ...)$.

This shows that the **logarithmic function** is infinitely many valued. For any particular value of n a unique branch of the function is determined, and the logarithm becomes effectively single valued. If $n = 0$, the resulting branch of the logarithm function is the principal value.

At all points except the points on the non-positive real-axis, each branch of $\ln z$ is continuous and analytic. In fact, from the definition

$$\ln\, z = \ln|z| + i \arg z = \frac{1}{2} \ln(x^2 + y^2) + i \tan^{-1}\frac{y}{z}.$$

One can verify that the $C - R$ equations are satisfied everywhere except at the origin.

Also, $$u = \frac{1}{2}\ln(x^2 + y^2) \text{ and } v = \tan^{-1}\frac{y}{x}$$

are continuous except on the non-positive real axis.

Hence

$$\begin{aligned}\frac{d}{dz}(\ln z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\\ &= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\\ &= \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}\end{aligned}$$

We now define the general power of a complex number $z = r(\cos\theta + i\sin\theta)$ by (31) $z^\alpha = e^{\alpha \ln z}$, ($\alpha$ complex, $z \neq 0$).

Since $\ln z$ is infinitely many valued, it follows that z^α is also infinitely many valued. We can write equation (31) as

$$\begin{aligned}z^\alpha &= \exp\{\alpha\,[\ln|z| + i(\theta + 2n\pi)]\}\\ &= \exp\{\alpha(\ln|z|)e^{i\alpha\theta}\, e^{2n\alpha\pi i}\end{aligned}$$

where θ is the principal value of the arg z.

3.7 INVERSE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

The function $w = \sin^{-1}z$ is defined as the function which satisfies the relation

$$\sin w = z.$$

Similarly, we can define $\cos^{-1}z$, $\tan^{-1}z$, $\cot^{-1}z$, $\sec^{-1}z$, $\operatorname{cosec}^{-1}z$, $\sinh^{-1}z$, $\cosh^{-1}z$, $\tanh^{-1}z$, $\coth^{-1}z$, $\operatorname{sech}^{-1}z$, and $\operatorname{cosech}^{-1}z$.

Note that the inverse trigonometric and hyperbolic functions can be expressed in terms of natural logarithms. We derive the following formula:

(32) $\sin^{-1} z = -i\ln(iz \pm \sqrt{1 - z^2})$

Let $w = \sin^{-1}z$. Then

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}$$

Hence, $$e^{2iw} - 2ize^{iw} - 1 = 0$$

Solving this equation as a quadratic in e^{iw}, we obtain

$$e^{iw} = iz \pm \sqrt{1 - z^2}$$

Taking logarithms and solving for w, we have

$$w = -i \ln (iz \pm \sqrt{1 - z^2}).$$

We now simply write the other formulas

(33) $\cos^{-1} z = -i \ln (z \pm \sqrt{z^2 - 1})$

(34) $\tan^{-1} z = \dfrac{i}{2} \ln \dfrac{i + z}{i - z}$

(35) $\cosh^{-1} z = \ln (z \pm \sqrt{z^2 - 1})$

(36) $\sinh^{-1} z = \ln (z \pm \sqrt{z^2 + 1})$

(37) $\tanh^{-1} z = \dfrac{1}{2} \ln \dfrac{1 + z}{1 - z}$.

EXERCISES

1. Let f be the function defined by

$$f(z) = \lim_{n \to \infty} \left(\frac{z^n - 1}{z^n + 1} \right).$$

Show that the limit exists for $|z| \neq 1$. Is it possible to define $f(z)$ when $|z| = 1$ in such a way to make f continuous?

2. Let f be the function defined by

$$f(z) = \lim_{n \to \infty} \frac{z^n}{1 + z^n}.$$

For what values of z the limit exists?

3. For what values of z are each of the following functions continuous?

(*i*) $f(z) = \dfrac{1}{z} + \dfrac{1}{z^2 + 1}$

(*ii*) $f(z) = (x + y^2) + ixy$

(*iii*) $f(z) = \begin{cases} \dfrac{z^2 + 3iz - 2}{z + i}, & z \neq -i \\ i, & z = -i \end{cases}$

(*iv*) $f(z) = \begin{cases} z^2 + iz + 2, & z \neq i \\ i, & z = i \end{cases}$

4. Show that the function

$$f(x, y) = \frac{x + iy}{x - iy}$$

is not continuous at the origin.

5. Show that the following functions are not differentiable

(i) $f(z) = |z|$ *(ii)* $f(z) = \text{Im}\, z$ *(iii)* $f(z) = \bar{z}$.

6. Use the Cauchy-Riemann equations and examine which of the following functions are differentiable

(i) $f(z) = x^2 + y^2$ *(ii)* $f(z) = \dfrac{1}{x + iy}$

(iii) $f(z) = e^y (\cos x + i \sin x)$.

At which points do the *C–R* conditions hold in each case?

7. Prove that the function

$$f(z) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} + i\dfrac{x^3 + y^3}{x^2 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

satisfies the $C - R$ equations at the origin but does not have a derivative.

8. Show that the $C - R$ equations in polar coordinates are given by

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \phi}, \frac{\partial u}{\partial \phi} = -r\frac{\partial v}{\partial r}$$

where

$$f(z) = f(r, \phi) = u(r, \phi) + iv(r, \phi).$$

9. If $f(z)$ and $g(w)$ are analytic functions, prove that $g(f(z))$ is analytic.

10. Prove that an analytic function cannot have a constant absolute value without reducing to a constant.

11. Let G be a region and define $G_1 = \{z : \bar{z} \in G\}$. If $f : G \to C$ is analytic, prove that $f_1 : G_1 \to C$ defined by $f_1(z) = \overline{f(\bar{z})}$ is analytic.

12. If $f(z)$ is an analytic function, prove that

(i) $$\left[\frac{\partial}{\partial x}|f(z)|\right]^2 + \left[\frac{\partial}{\partial y}|f(z)|\right]^2 = |f'(z)|^2$$

(ii) $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2.$$

13. Show that the following functions are harmonic and find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$.

(i) $u = \arg z$ *(ii)* $u = e^x \cos y$ *(iii)* $u = x^2 - y^2$.

14. If u and v are harmonic in a region Ω, prove that

$$\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \text{ is analytic in } \Omega.$$

15. Prove that a harmonic function satisfies the differential equation

$$\frac{\partial^2 u}{\partial z\, \partial \bar{z}} = 0.$$

16. Let Ω be a region which is a symmetric with respect to the real axis; that is, when $z \in \Omega$. Also $\bar{z} \in \Omega$. Show that if f is analytic on $\Omega_1 = \Omega \cap \{z : \text{Im}\, z > 0\}$, then the function ϕ with values $\phi(z) = \overline{f(\bar{z})}$ is analytic on $\Omega_2 = \Omega \cap \{z : \text{Im}\, z < 0\}$.

17. Let $U(x, y) = x/(x^2 + y^2)$, $(x, y) \in R^2 \sim \{(0, 0)\}$. By using the $C - R$ equations, find a function $V(x, \mathrm{y})$ such that if $u(z) = u(x + iy) = U(x, y)$ and $v(z) = v(x + iy) = V(x, y)$, the function $f = u + iv$ is analytic on $C \sim \{0\}$.

18. Prove that there cannot exist a function analytic on an open set $E \subset C$ with real part $x - 2y^2$.

19. Find the principal value of the following functions

(*i*) $(1 + i)^{2 - i}$

(*ii*) $(1 - i)^{2 - 3i}$

(*iii*) $\tan i$

(*iv*) $\ln(-3 + 4i)$

(*v*) 2^i.

20. Show that there is no branch of the logarithm defined on $G = C \sim \{0\}$.

4

LINE INTEGRAL AND CAUCHY'S THEOREM

4.1 DEFINITIONS

Let $[a, b]$ be a closed interval. A set of points $P = \{t_0, t_1, t_2, ..., t_n\}$ satisfying $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$ is called a **partition** of $[a, b]$. The interval $[t_{k-1}, t_k]$ is called the kth **sub-interval** of P so that

$$\sum_{k=1}^{n} (t_k - t_{k-1}) = b - a.$$

Let $\Omega \subset C$ be a region. Let $[a, b]$ be some interval. A path in Ω is defined to be a continuous function

$$\gamma : [a, b] \to \Omega.$$

We call $\gamma(a)$ the **initial point,** and $\gamma(b)$ the **terminal point.**

We call γ a **differentiable path** if $\gamma'(u)$ exists for each $u \in [a, b]$ and $\gamma' : [a, b] \to C$ is continuous. γ is said to be **piecewise differentiable** if there is a partition of $[a, b]$ such that γ is differentiable on each sub-interval

$$[t_{t-1}, t_i], 1 \le i \le n.$$

Note that a function $\gamma : [a, b] \to C$ has a derivative $\gamma'(u)$ for each $u \in [a, b]$ implies that

$$\lim_{h \to 0} \frac{\gamma(u+h) - \gamma(u)}{h} = \gamma'(u)$$

exists for $a < u < b$ and the right and left limits exist for

$$u = a \text{ and } u = b.$$

Note also that in the definition of differentiable path the continuity of γ' is included. Many authors call this "**continuously differentiable**" path.

Let $\gamma : [a, b] \to C$. Let $P = \{t_0, t_1, t_2, ..., t_n\}$ be a partition of $[a, b]$. Write

$$\Delta\gamma_k = \gamma(t_k) - \gamma(t_{k-1}), k = 1, 2, ..., n.$$

If there is constant $M > 0$ such that

$$\nu(\gamma, P) = \sum_{k=1}^{n} |\Delta\gamma_k| \leq M$$

for all partitions of $[a, b]$, then γ is said to be of **bounded variation** of $[a, b]$.

By total variation of γ, denoted by $V(\gamma)$, we mean

$$V(\gamma) = \sup \{\nu(\gamma, P) : P \text{ is a partition of } [a, b]\}. \text{ Note that}$$

$$V(\gamma) \leq M < \infty.$$

A partition Q of $[a, b]$ is called a **refinement** of P if $Q \supset P$.

We state one useful formula regarding the variation of g when g' is continuous on $[a, b]$.

(1) $V(g) = \int_a^b |g'| \, dx.$

Properties

(*i*) γ is of bounded variation iff Re γ and Im γ are of bounded variation.

(*ii*) If γ is real valued and non-decreasing, then γ is of bounded variation and

$$V(\gamma) = \gamma(b) - \gamma(a).$$

(*iii*) Let $\gamma : [a, b] \to C$ be of bounded variation. If P and Q are partitions of $[a, b]$ and $P \subset Q$, then

$$\nu(\gamma, P) \leq \nu(\gamma, Q).$$

We leave it to the reader to verify the validity of these properties.

4.2 RIEMANN-STIELTJES INTEGRAL

In order to define the integral of a function along a path in C we need the notion of Riemann-Stieltjes integral. For detailed discussion on Riemann-Stieltjes integral we refer to books on Real Analysis by Rudin, Apostol and Bartle.

Let $f, g : [a, b] \to C$ be bounded functions. Corresponding to every partition $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$ we choose points $u_0, u_1, ..., u_{n-1}$ such that

$$x_k \leq u_k \leq x_{k+1}, \; k = 0, 1, 2, ..., n-1.$$

Introduce

$$S(P, f, g) = \sum_{k=0}^{n-1} f(u_k) [g(x_{k+1}) - g(x_k)]$$

and consider the behaviour of $S(P, f, g)$ when the norm of P is small.

We say that f is Riemann-Stieltjes integrable with respect to g provided that there exists a number $A \in C$ such that for each positive η there exists a positive δ such that

$$|S(P, f, g) - A| < \eta$$

when $\| P \| < \delta$. [(11.11) is defined in Exercise 20 of Chapter 2].

This is true for all P and all admissible choices of u_k. It is clear that there is atmost one such A. In case such A exists, we call A the Riemann-Stieltjes integral of f with respect to g and denote it by

$$\int_a^b f(x)\,dg(x) \quad \text{or} \quad \int_a^b f\,dg.$$

We have already defined a path as a continuous function $\gamma : [a, b] \to C$. The set $\{\gamma(t) : a \le t \le b\}$ is called the trace of γ where $\gamma : [a, b] \to C$ is a path. Denote the trace of γ by $\{\gamma\}$. Note that the set $\{\gamma\}$ is compact. Denote the length of γ by $L(\gamma)$.

A path γ is said to be **rectifiable** if $L(\gamma) < +\infty$. If γ is piecewise differentiable, then γ is rectifiable and

$$L(\gamma) = \int_a^b |\gamma'|\,dt.$$

The following fact will be useful while defining the line-integral.

Let $\gamma : [a, b] \to C$ be a **rectifiable path** with $\{\gamma\} \subset \Omega \subset C$ and let $f : \Omega \to C$ be continuous function. Then $f \circ \gamma$ (composite function) is also continuous on $[a, b]$.

4.3 LINE-INTEGRAL

Let $\gamma : [a, b] \to C$ be rectifiable. Let f be defined and continuous on the trace of γ. Then the **line-integral** of f along γ is defined by the expression

$$\int_a^b f(\gamma(t))\,d\gamma(t) = \int_a^b (f \circ \gamma)\,d\gamma.$$

Denote this integral by

$$\int_\gamma f(z)\,dz = \int_\gamma f\,dz.$$

Example 1: Let $\gamma : [0, 2\pi] \to C$ be given by

$$\gamma(\theta) = e^{i\theta}$$

and define $$f(z) = \frac{1}{z} \text{ for } z \neq 0.$$

We have to evaluate the integral of f over the circle $\{\gamma\}$ *i.e.*

$$\int_\gamma \frac{1}{z} dz \ (0 \le \theta \le 2\pi).$$

By definition, this integral is equal to

$$\int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta}\,d\theta = i\int_0^{2\pi} d\theta = 2\pi i.$$

Exercise: Let $\gamma : [a, b] \to C$ be a rectifiable curve. Define the reverse or opposite curve to be

$$-\gamma : [a, b] \to C$$

such that

$$-\gamma(t) = \gamma(a + b - t).$$

Let f be continuous on $\{\gamma\}$. Show that

(*i*) $\int_{-\gamma} f = -\int_{\gamma} f$.

(*ii*) $\left|\int_{\gamma} f\right| \le \int_{\gamma} |f|\,|dz| \le V(\gamma) \sup [|f(z)| : z \in \{\gamma\}]$.

We shall prove one important theorem for the line-integral which is analogous to the **Fundamental Theorem of Calculus.** We will need the concept of **modulus of continuity.**

4.4 MODULUS OF CONTINUITY

Let (X, d_1) and (Y, d_2) be metric spaces. Recall that f is uniformly continuous provided that for each positive ε there exists a positive δ such that

$$d_2(f(x), f(y)) < \varepsilon$$

when $d_1(x, y) < \delta$.

Consider the set

(2) $\{d_2(f(x), f(y)) : d_1(x, y) \le \delta\}$

where $\delta \ge 0$.

When this set is bounded, denote its supremum by $w_f(\delta)$. The function w_f assigning to each such δ the value $w_f(\delta)$ is called the modulus of continuity of f.

The expression (2) on the notion of uniform continuity is given by dela Vallée Poussin.

Note that f is uniformly continuous iff the domain of w_f does not reduce to 0 and w_f is continuous at 0 $(X \ne \phi)$.

Theorem 1. *Let $f : \Omega \to C$ where Ω is a region in C and f is continuous on Ω. Let $\gamma : [a, b] \to \Omega$ be rectifiable. Suppose that g is a primitive of f. Then*

(3) $\int_{\gamma} f\,dz = g[\gamma(b)] - g[\gamma(a)]$

(*g is a primitive of f when g′ = f*).

Proof: *Case I.* Suppose γ' is continuous on $[a, b]$. Then

$$\int_{\gamma} f\,dz = \int_a^b (g \circ \gamma)'\,dt$$

and hence (3) holds.

If the domain of γ is [0, 1], then replace γ by $\gamma\,[a + (b - a)t]$, $0 \le t \le 1$.

Case II. Suppose Ω is an open disk and γ is taken arbitrary. Let

$$\Omega = D_r(c) = \{z : |z - c| < r\}.$$

Since $\{\gamma\}$ is compact we have

$$|\gamma(t) - c| \le \rho < r,\ 0 \le t \le 1$$

for some ρ, $0 < \rho < 1$.

Let P be a partition of $[0, 1]$ and $\{t_0, t_1, ..., t_n\}$ its associated sequence. Define

$$\Gamma : [a, b] \to C \text{ by}$$

$$\Gamma(t) = \gamma(t_k)\,\frac{t_{k+1} - t}{t_{k+1} - t_k} + \gamma(t_{k+1})\,\frac{t - t_k}{t_{k+1} - t_k}$$

$$t_k \le t \le t_{k+1},\ k = 0, 1, ..., m - 1.$$

Now
$$\int_\Gamma f\,dz - \sum_0^{m-1} f[\gamma(t_k)]\ \{\gamma(t_{k+1}) - \gamma(t_k)\}$$

$$= \sum_0^{m-1} \int_{t_k}^{t_{k+1}} [f \circ \Gamma - (f \circ \Gamma)(t_k)]\,d\Gamma.$$

But Γ is a polygon so, from case I

$$\int_\Gamma f\,dz = g[\gamma(1)] - g[\gamma(0)].$$

Therefore

$$(4)\ |\ \{g[\gamma(1)] - g[\gamma(0)]\} - \sum_0^{m-1} f[\gamma(t_k)]\{\gamma(t_{k+1}) - \gamma(t_k)\}\ |$$

$$\le w_F\,[w_\gamma(\|P\|)]\,L\,[\gamma]$$

where $F = f \,|\, \overline{D}_\rho\,(c)$ and w_F and w_γ are the modulus of continuity of the functions.

Thus (3) follows from (4) and the theorem is proved for Γ restricted to each $[t_k, t_{k+1}]$.

Case III. The general case

Consider $\Omega \ne C$. Since $\{\gamma\}$ is compact, we have

$$\min \{|\,\gamma(t) - w\,| : 0 \le t \le 1,\ w \in C - \Omega\} > 0.$$

Denote the above expression by μ. Let $n \in N$ (the set of natural numbers) be so large that

$$w_\gamma(2^{-n}) < \mu.$$

Then

$$|\,\gamma(t) - \gamma(t_k)\,| < \mu,\ t_k \le t \le t_{k+1}$$

$$k = 0, ..., 2^n - 1, \text{ where } t_k = \frac{k}{2^n}.$$

Hence, the integral of f along $\gamma[t_k, t_{k+1}]$ is

$$g[\gamma(t_{k+1})] - g[\gamma(t_k)]$$

and summing we obtain (3). This completes the proof of the theorem.

A curve $\Gamma : [a, b] \to C$ is said to be closed if $\gamma(a) = \gamma(b)$. The following result is an immediate consequence of Theorem 1.

Theorem 2. *Let Ω be open in C and let γ be rectifiable in Ω. Let $f : \Omega \to C$ be continuous with a primitive $g : \Omega \to C$. If γ is closed curve then*

$$\int_\gamma f = 0.$$

Example: Let $f(z) = z^n$, where n is an integer $\neq -1$. Then for any closed path γ, we have

$$\int_{\gamma} z^n \, dz = 0.$$

If n is negative, the result holds good for any closed path not passing through the origin. This is true because z^n has the primitive $\dfrac{z^{n+1}}{n+1}$.

We know from the Fundamental Theorem of calculus that each continuous function has a primitive. But this is not true for functions of a complex variable. For example, let

$$f(z) = |z|^2 = x^2 + y^2.$$

If g is a primitive of f then g is analytic. Write $g = \xi + i\eta$. Then

$$x^2 + y^2 = g'(x + iy)$$

By $C - R$ equations

$$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y} = x^2 + y^2$$

and $$\frac{\partial \xi}{\partial y} = -\frac{\partial \eta}{\partial x} = 0.$$

But $\dfrac{\partial \xi}{\partial y} = 0$ implies that $\xi(x, y) = \psi(x)$ for some differentiable function ψ.

Thus $x^2 + y^2 = \dfrac{\partial \xi}{\partial x} = \psi'(x)$, which is a contradiction. This shows that $f(z) = |z|^2$ does not have a primitive.

4.5 LOCAL PRIMITIVE

We would like to define a **primitive** for f on some open disc centred at z_0. Before finding a **primitive locally** we prove **Cauchy's theorem** for a rectangle. The proof is due to E. Goursat. We denote by T the **rectangle** and by δT the **boundary of the rectangle** (the path describing the boundary of the rectangle will be taken counterclockwise). Note that T is meant as the set of points inside and on the boundary of the rectangle, so T is not a region. In the following theorem we consider a function f which is analytic on a region which contains T.

Theorem 3 (Cauchy-Goursat). *Let T be a rectangle, and let f be analytic on T. Then*

$$\int_{\delta T} f = 0.$$

Proof: Join the mid-points of the sides of the rectangle T to obtain four rectangles T_1, T_2, T_3, T_4 as shown in Fig. 4.I.

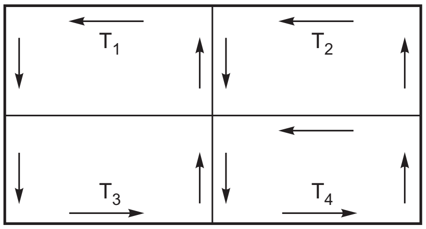


Fig. 4.I

Then $$\int_{\delta T} f = \sum_{i=1}^{4} \int_{\delta T_i} f.$$

Hence $$\left| \int_{\delta T} f \right| \le \sum_{i=1}^{4} \left| \int_{\delta T_i} f \right|.$$

It follows that there is one rectangle, say $T^{(1)}$, among T_1, T_2, T_3 and T_4 such that

$$\left| \int_{\delta T^{(1)}} f \right| \ge \frac{1}{4} \left| \int_{\delta T} f \right|.$$

Next, join the mid-points of the sides of the rectangle $T^{(1)}$ to obtain four rectangles as shown in Fig. 4.II.

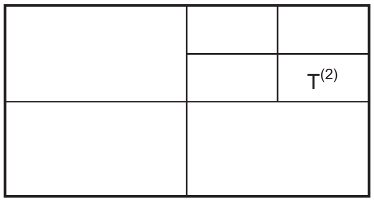


Fig. 4.II

We denote one of the four rectangles thus obtained by $T^{(2)}$ and have the similar inequality.

$$\left| \int_{\delta T^{(2)}} f \right| \ge \frac{1}{4} \left| \int_{\delta T^{(1)}} f \right|.$$

By repeating this process, we obtain a sequence of rectangles

$T^{(1)} \supset T^{(2)} \supset T^{(3)} \supset T^{(4)}$ ... such that

$$\left| \int_{\delta T^{(n+1)}} f \right| \ge \frac{1}{4} \left| \int_{\delta T^{(n)}} f \right|.$$

Hence $$\left| \int_{\delta T^{(n)}} f \right| \ge \frac{1}{4^n} \left| \int_{\delta T} f \right|.$$

Let L_n be the length of $\delta T^{(n)}$. Then

$$L_{n+1} = \frac{1}{2} L_n.$$

It follows by induction that

$$L_n = \frac{1}{2^n} L_0,$$

where L_0 is the length of δT.

Note that the diameter of $T^{(n)}$ tends to 0 as $n \to \infty$. It follows from Cantor's theorem that the intersection $\bigcap_{n=1}^{\infty} T^{(n)}$ consists of a single point z_0. Since f is complex differentiable at z_0, there is a disk $D_r(z_0)$ such that for all $z \in D_r(z_0)$ we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\,\eta(z, z_0)$$

where
$$\lim_{z \to z_0} \eta(z, z_0) = 0.$$

It follows that $T^{(n)}$ is contained in $D_r(z_0)$ for sufficiently large n and we have

$$(5) \quad \int_{\delta T^{(n)}} f(z)\,dz = \int_{\delta T^{(n)}} f(z_0)\,dz + f'(z_0)\int_{\delta T^{(n)}} (z - z_0)\,dz + \int_{\delta T^{(n)}} (z - z_0)\,\eta(z, z_0)\,dz.$$

The first two integrals on the right side of (5) are zero.

Hence
$$\int_{\delta T^{(n)}} f = \int_{\delta T^{(n)}} (z - z_0)\,\eta(z, z_0)\,dz$$

and we obtain the inequality

$$\frac{1}{4^n}\left|\int_{\delta T} f\right| \le \left|\int_{\delta T^{(n)}} f\right| = \left|\int_{\delta T^{(n)}} (z - z_0)\,\eta(z, z_0)\,dz\right| \le \frac{1}{4^n} L_0 \operatorname{diam}(T^{(n)}) \sup|\eta(z, z_0)|$$

where the sup is taken for all $z \in T^{(n)}$. Thus

$$(6) \quad \left|\int_{\delta T} f\right| \le L_0 \operatorname{diam}(T^{(n)}) \sup|\eta(z, z_0)|.$$

The right side of (6) tends to 0 as $n \to \infty$. Hence

$$\int_{\delta T} f = 0$$

and this completes the proof of the theorem.

We now find a primitive locally.

Theorem 4. *Let f be continuous on $D_r(z_0)$. Let*

$$\int_{\delta T} f = 0$$

for each $T \subset D_r(z_0)$. Define

$$F(z_1) = \int_{z_0}^{z_1} f$$

where z_1 *is an arbitrary point in the disc* $D_r(z_0)$ *and the integral is taken along the sides of a rectangle T whose opposite vertices are* z_0 *and* z_1. *Then F is analytic on* $D_r(z_0)$ *and*

$$F'(z) = f(z).$$

Proof: We have

$$F(z_1 + h) - F(z_1) = \int_{z_1}^{z_1+h} f(z)\, dz.$$

The integral between z_1 and $z_1 + h$ is taken as shown in Fig. 4.III.

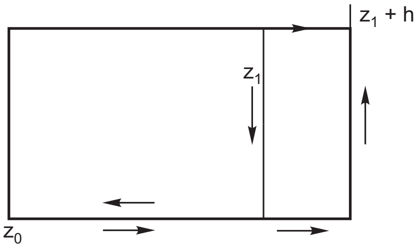


Fig. 4.III

Since f is continuous at z_1, there is a function $\phi(z)$ such that

$$f(z) = f(z_1) + \phi(z)$$

where $\lim_{z \to z_1} \phi(z) = 0$.

Then $$F(z_1 + h) - F(z_1) = \int_{z_1}^{z_1+h} f(z_1)\, dz + \int_{z_1}^{z_1+h} \phi(z)\, dz.$$

$$= hf(z_1) + \int_{z_1}^{z_1+h} \phi(z)\, dz.$$

Dividing both sides by h we have

$$(7)\quad \frac{F(z_1 + h) - F(z_1)}{h} = f(z_1) + \frac{1}{h}\int_{z_1}^{z_1+h} \phi(z)\, dz.$$

The length of the path from z_1 to $z_1 + h$ is bounded by $|h_1| + |h_2|$.

Hence

$$(8)\quad \left| \frac{1}{h}\int_{z_1}^{z_1+h} f(z)\, dz \right| \le \frac{1}{|h|}(|h_1| + |h_2|) \sup \phi(z).$$

Note that the sup is taken for z on the path of integration. The expression on the right side of (8) tends to 0 as $z \to z_1$.

Hence

$$\lim_{h \to 0} \frac{F(z_1 + h) - F(z_1)}{h} = f(z_1).$$

This proves the theorem.

It now follows from the above discussion that if f has a primitive on a disc $D_r(z_0)$, then the integral of f along any path between z_0 and z in $D_r(z_0)$ is independent of the path. Thus according to Theorem 2 in the previous section we find.

Theorem 5. *If f is analytic on a disc D, then f has a primitive on D, and the integral of f along any closed path in D is 0.*

4.6 CAUCHY'S THEOREM (HOMOTOPY FORM)

We first give the general definition of **homotopy.**

Definition: Let Ω be an open set. Let γ_0, γ_1 be two closed paths in Ω. Assume that they are defined on the same interval [0, 1]. We say that γ_0 is **homotopic** to γ_1 in Ω if there exists a continuous function

$$\psi : [0, 1] \times [0, 1] \to \Omega$$

such that

(*i*) $\psi(t, 0) = \gamma_0(t)$, $\psi(t, 1) = \gamma_1(t)$ $(0 \le t \le 1)$;

(*ii*) $\psi(0, u) = \psi(1, u)$ $(0 \le u \le 1)$.

Thus, if we define

$$\gamma_u : [0, 1] \to \Omega \text{ by}$$
$$\gamma_u(t) = \psi(t, u)$$

then each γ_u is a closed curve and they form a continuous family of curves which start at γ_0 and go to γ_1.

We now give another definition of homotopy which is less general than the definition given above. We say that the curve γ is piecewise smooth if $\gamma(t)$ possesses a continuous derivative $\gamma'(t)$ everywhere except at a finite number of points of [0, 1] and $\gamma'(t) = x'(t) + iy'(t) \neq 0$ for all t, $0 \le t \le 1$.

Definition: Let $\gamma_0 = \psi(t, 0)$ and $\gamma_1 = \psi(t, 1)$ belong to the family of piecewise smooth closed curves in a region Ω. We say that γ_0 is homotopic to γ_1 in Ω if there exists a continuous function $\psi = \psi(t, u)$ such that

(*i*) ψ has continuous partial derivatives

$$\frac{\partial \psi}{\partial t} \neq 0, \ \frac{\partial \psi}{\partial u} \neq 0, \text{ except at a finite number of points.}$$

(*ii*) ψ maps the square

$$I^2 = \{(t, u) : 0 \le t \le 1, 0 \le u \le 1\} \text{ into } \Omega.$$

This definition will be used.

If γ_0 is homotopic to γ_1 with respect to Ω, and γ_1 is a point, then γ_0 is **homotopic to a point** with respect to Ω.

Note that if γ_0 and γ_1 are homotopic in a region Ω then they are homotopic in any region Ω_1 containing Ω.

Remark: Suppose that two piecewise smooth, closed curves γ_0 and γ_1 and a region Ω are given. In order to exhibit the homotopy between γ_0 and γ_1 we have to find a family of piecewise smooth, closed curves $\psi(t, u)$ such that (*i*) γ_0 and γ_1 belong to this family where $\gamma_0(t) = \psi(t, 0)$ and $\gamma_1(t) = \psi(t, 1)$.

(*ii*) $\psi : I^2 \to C$ is continuous with $\psi(I^2) \subset \Omega$ such that $\psi(t, u)$ piecewise smooth closed curve for every u and $\psi(t, u)$ is piecewise smooth closed curve for every t.

From an intuitive standpoint the notion of homotopy is simple. This means that we can transform γ_0 into γ_1 by continuous deformations given by $\psi(t, u)$. Note that during the process of deformation $\psi(t, u)$, $0 \leq u \leq 1$, does not leave Ω. We illustrate this notion in Fig. 4.IV.

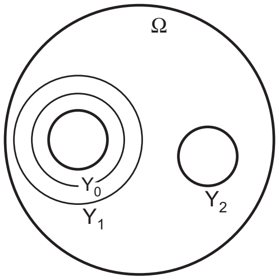


Fig. 4.IV

In Fig. 4.IV, γ_0 and γ_1 are homotopic in Ω but γ_0 and γ_2 are not homotopic in Ω. γ_2 is homotopic to a point in Ω.

Examples: (*i*) Let $\gamma_0(t) = \frac{1}{2}(\cos 2\pi t + i \sin 2\pi t)$, $0 \leq t \leq 1$ and $\gamma_1(t) = \cos 2\pi t + i \sin 2\pi t$, $0 \leq t \leq 1$ be two circles. Then γ_0 and γ_1 are homotopic in any region Ω containing the closed disc $\overline{D}(0, 1)$.

Take $$\psi(t, u) = \frac{1+u}{2}(\cos 2\pi t + i \sin 2\pi t).$$

Then $$\psi(t, 0) = \gamma_0(t)$$

and $$\psi(t, 1) = \gamma_1(t).$$

Since $|\psi(t, u)| \leq 1$ for $0 \leq u \leq 1$, it follows that

$$\psi(t, u) \subset \overline{D}(0, 1) \subset \Omega.$$

Also $$\psi(0, u) = \psi(1, u).$$

So the curves γ_0 and γ_1 are homotopic in Ω containing $\overline{D}(0, 1)$.

(*ii*) Let Ω be a convex set, and let γ_0, γ_1 be two piecewise smooth closed curves in Ω. Then γ_0 and γ_1 are homotopic in Ω.

Define

$$\psi(t, u) = u\gamma_0(t) + (1 - u)\gamma_1(t).$$

It can be checked that $\psi = \psi(t, u)$ is continuous and each curve $\psi(t, u)$ is a closed curve. Also,

$$\psi(t, 0) = \gamma_1(t)$$

and $$\psi(t, 1) = \gamma_0(t).$$

In order to simplify the proof of the homotopy form of Cauchy's theorem, we introduce the notion of "**near-together**". Let Ω be an open set. Let γ_0 and γ_1 be defined on the same interval [0, 1]. We say that γ_0 and γ_1 are near-together if there is a partition

$$0 = u_0 \le u_1 \le u_2 \le \ldots \le u_n = 1$$

and for each $j = 0, 1, 2, \ldots, n - 1$ there exists a disc $D_j \subset \Omega$ such that the images of each segment under the mappings of γ_0 and γ_1 are contained in D_j. In other words

$$\gamma_0([u_j, u_{j+1}]) \subset D_j$$

and $$\gamma_1([u_i, u_{j+1}]) \subset D_j.$$

Lemma: *Let γ_0 and γ_1 be two piecewise smooth closed curves in the open set Ω. Let γ_0, γ_1 be defined on the same interval [0, 1]. Assume that they are near-together. Then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

Proof: Choose a partition

$$0 = u_0 \le u_1 \le \ldots \le u_n = 1$$

and for each $j = 0, 1, 2, \ldots, n - 1$, choose discs $D_j \subset \Omega$. Let F_j be a primitive of f on D_j. Let

$$z_j = \gamma_0(u_j) \quad \text{and} \quad w_j = \gamma_1(u_j).$$

Then

$$\int_{\gamma_0} f - \int_{\gamma_1} f = \sum_{j=0}^{n-1} [F_j(z_{j+1}) - F_j(z_j) - (F_j(w_{j+1}) - F_j(w_j))]$$
$$= \sum_{j=0}^{n-1} [F_j(z_{j+1}) - F_j(w_{j+1}) - (F_j(z_j) - F_j(w_j)]$$

i.e.

$$(9) \quad \int_{\gamma_0} f - \int_{\gamma_1} f = F_{n-1}(z_n) - F_{n-1}(w_n) - (F_0(z_0) - F_0(w_0)).$$

Since γ_0 and γ_1 are closed, we have $z_0 = z_n$ and $w_0 = w_n$. Let there be some disc D which contains z_0 and w_0. The primitives F_{n-1} and F_0 differ by a constant on $D \subset \Omega$. Hence the right-side of (9) is equal to 0. This proves the lemma.

Theorem 6 (Cauchy's Theorem). *Let Ω be an open set. Let γ_0 and γ_1 be two piecewise smooth closed curves in Ω, and assume that they are homotopic in Ω. Let f be analytic on Ω. Then*

$$\int_{\gamma_0} f - \int_{\gamma_1} f.$$

In particular, if γ_0 is homotopic to a point in Ω, then

$$\int_{\gamma_0} f = 0.$$

Proof: Let $\psi : [0, 1] \times [0, 1] \to \Omega$ be the homotopy. Since ψ is continuous and I^2 is compact, ψ is uniformly continuous and $\psi(I^2)$ is a compact subset of Ω. By uniform continuity we can find partitions

$$0 = u_0 \le u_1 \le \dots \le u_n = 1$$

$$0 = t_0 \le t_1 \le \dots \le t_n = 1$$

of these intervals, such that if

S_{ik} = small square $[u_i, u_{i+1}] \times [t_k, t_{k+1}]$, then the image $\psi(S_{ik})$ is contained in a disc D_{ik} which is itself contained in Ω.

Let ψ_k be the continuous curve defined by

$$\psi_k(t) = \psi(t, u_k),\ k = 0, 1, \dots, m.$$

Then the continuous curves ψ_k, ψ_{k+1} are near-together. By Lemma we have

$$\int_{\psi_k} f = \int_{\psi_{k+1}} f.$$

Since $\psi_0 = \gamma_0$ and $\psi_m = \gamma_1$, we conclude that

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

This proves the theorem.

Definition: We say that an open set Ω is **star shaped** if there is a point z_0 in Ω such that for each z in Ω the line segment $[z_0, z]$ lies entirely in Ω.

Note also that each star shaped set is connected.

We now define **simply connected set.**

Definition: We say that an open set Ω is simply connected if it is connected and every closed path in Ω is homotopic to a point.

We are now in a position to construct primitives for a wider class of open sets.

4.7 GLOBAL PRIMITIVES

Theorem 7. *Let Ω be a simply connected open set. Let f be analytic on Ω. Define*

$$F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta$$

where z is any point in Ω and z_0 is a fixed point in Ω. Then

(*i*) *$F(z)$ is independent of the path in Ω from z_0 to z.*

(*ii*) $F'(z) = f(z)$.

Proof: Since Ω is open and connected there always exist a path from z_0 to any other point of Ω. Let γ_0 and γ_1 be two paths in Ω from z_0 to a point z in Ω. Let $-\gamma_1$ be the opposite (reverse) path of γ_1 from z to z_0. Then

$$\gamma = \{\gamma_0, -\gamma_1\} \text{ is a closed path.}$$

Hence, by Theorem 6

$$0 = \int_{\gamma} f = \int_{\gamma_0} f + \int_{-\gamma_1} f$$

i.e.

$$\int_{\gamma_0} f - \int_{\gamma_1} f = 0.$$

This proves (*i*).

In order to prove (*ii*) we need to prove the differentiability of F at a point z_1. Let $z_1 \in \Omega$ and $\rho > 0$ be such that $D(z_1, \rho) \subset \Omega$. If z is in $D(z_1, \rho)$, then we may choose a path z_0 to z by passing through z_1. Hence

$$(10)\ F(z) = F(z_1) + \int_{z_1}^{z} f.$$

It follows from Theorem 4 that the integral in the right side of (10) defines a local primitive for f in a neighbourhood of z_1. Hence

$$F'(z) = f.$$

Examples (*i*) Every star shaped region is simply connected.

(*ii*) Let $\Omega = C - \{\theta e^{i\theta} : 0 \leq \theta < \infty\}$.

Then Ω is simply connected.

(*iii*) The domain of the principal branch of the logarithm is simply connected.

EXERCISES

1. Let $\gamma(t) = \exp\{(-1 + i)t^{-1}\}$ for $0 < t \leq 1$ and $\gamma(0) = 0$. Prove that γ is a rectifiable path and find $V(\gamma)$. Sketch the trace of γ.

2. Define $\gamma : [0, 1] \to C$ by

$$\gamma(t) = \begin{cases} t + it \sin \dfrac{1}{t}, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

Prove that γ is a path but is not rectifiable. Sketch this path.

3. Let γ_1 and γ_2 be the two polygons $[1, i]$ and $[1, 1 + i, i]$. Express γ_1 and γ_2 as paths and evaluate $\int_{\gamma_1} f(z)\, dz$ and $\int_{\gamma_2} f(z)\, dz$ where $f(z) = |z|^2$.

4. Show that the function $f(z) = \bar{z}$ does not have a primitive on $E = \{z : \operatorname{Im} z \neq 0\}$.

5. Let γ be a path in C with terminal point z_1 and initial point z_0. Let $a \neq 0$ and b be complex numbers. Find $\int_{\gamma} e^{az + b}\, dz$.

6. Let γ be any closed path in C. Show that $\int_{\gamma} \exp(-z^2)\, dz = 0$.

7. Let f be continuous on a region G. Show that if for any two points $a \in G$, $b \in G$, the integral $\int_{\gamma} f(z)\, dz$ is independent of the path γ joining a to b, then $\int_{\gamma_0} f(z)\, dz = 0$ for every closed path γ_0 in G.

8. Find the integral $\int_\gamma \sin z \; dz$ from the origin to the point $1 + i$, taken along the parabola $y = x^2$.

9. Let f be analytic in a region that contains γ. Show that $\int_\gamma \overline{f(z)} \; f' \; dz$ is purely imaginary. (The continuity of $f'(z)$ is taken for granted).

10. Let f be analytic and satisfies the inequality $| f(z) - 1 | < 1$ in a region Ω. Show that $\int_\gamma \frac{f'(z)}{f(z)} \, dz = 0$ for every closed curve γ in Ω. (The continuity of $f'(z)$ is taken for granted).

11. Let $P(z)$ be a polynomial and let C_1 be the circle $| z - a | = r$. Evaluate $\int_{C_1} P(z) \, d\bar{z}$.

12. Let γ be the closed polygon $[1 - i, 1 + i, - 1 - i, 1 - i]$. Find $\int_\gamma \frac{dz}{z}$.

13. Let $\gamma(t) = 2e^{it}$ for $- \pi \le t \le \pi$. Evaluate $\int_\gamma (z^2 - 1)^{-1} \; dz$.

14. Evaluate $\int_0^{1+i} (z^2 + z) \, dz$. By choosing two different paths of integration show that the results are the same.

15. Show that $\left| \int_2^i (z + 1)^2 \; dz \right| \le 9\sqrt{5}$.

(**Hint:** Take as the path of integration the line $z = i + (2 - i) \, t$, $0 \le t \le 1$ and determine max $| (z + 1)^2 |$ on this line.

16. Evaluate $\int_1^{2+i} (x^2 + iy) \, dz$ along

(*i*) the line $y = x - 1$

(*ii*) the line $y = (x - 1)^2$.

17. Compute $\int_\gamma \frac{dz}{z^2 + 1}$ where $\gamma(\theta) = 2 \, | \cos 2\theta | \, e^{i\theta}$, $0 \le \theta \le 2\pi$.

18. Compute $\int_\gamma \frac{dz}{z^2 + \pi^2}$ where $\gamma(\theta) = \begin{cases} \theta e^{i\theta}, & 0 \le \theta \le 2\pi \\ 4\pi - \theta, & 2\pi \le \theta \le 4\pi \end{cases}$.

19. Show that

$$\left| \int_\gamma (x^2 - iy^2) dz \right| \le 2.5$$

where

(*i*) γ is the interval $[- i, i]$ on the y-axis.

(*ii*) γ is semi-circle $z = \cos \phi + i \sin \phi$, $- \frac{\pi}{2} \le \phi \le \frac{\pi}{2}$.

20. Show that

$$\left| \int_\gamma \frac{dz}{z^2 + 1} \right| \le 4\pi/3$$

where γ is the circle $\gamma(\varphi) = 2(\cos \varphi + i \sin \varphi)$, $0 \le \varphi \le 2\pi$.

21. Let γ be any closed path with graph not containing zero. Find

$$\int_\gamma \frac{\sin z}{z^2}\,dz.$$

22. Let $\{\gamma_n\}$ be the sequence of arcs,

$$\gamma_n(t) = \left(1 - \frac{1}{n+1}\right) + \frac{2t}{n+1} + it,\ 0 \le t \le 1,\ n = 1, 2, \ldots, \text{ and}$$

$$f(z) = z^2 + z + 1.$$

Find

$$\lim_{n \to \infty} \int_{\gamma_n} f(z)\,dz.$$

5

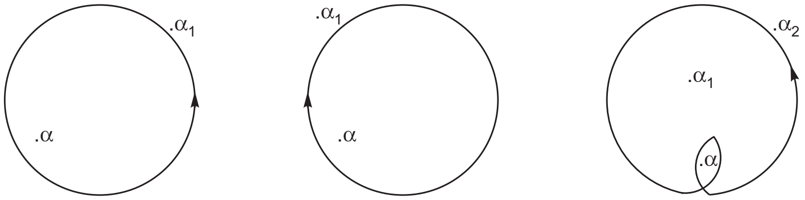
Application of Cauchy's Theorem

There are several versions of Cauchy's theorem. In this chapter, we first state the **homological version** of Cauchy's theorem and postpone the proof to a later chapter (see Appendix II). By using Cauchy's theorem we derive Cauchy's formula. In fact, Cauchy's formula will be quite sufficient for many applications. By paths we mean rectifiable curves. Circles are assumed oriented counterclockwise unless otherwise specified.

5.1 THE WINDING NUMBER

Let γ be a closed path in C and let $\alpha \notin \{\gamma\}$. Then **the index (the winding number)** of γ with respect to α denoted by $n(\gamma, \alpha)$, indicates how many times γ winds around α, and in which sense. It follows that $n(\gamma, \alpha)$ is a positive integer if γ winds counterclockwise around α and a negative integer if γ winds clockwise around α, $n(\gamma, \alpha) = 0$ if α is outside γ.

Example



Example I

$n(\gamma, \alpha) = 1$	$n(\gamma, \alpha) = -1$	$n(\gamma, \alpha) = 2$
$n(\gamma, \alpha_1) = 0$	$n(\gamma, \alpha_1) = 0$	$n(\gamma, \alpha_1) = 1$
		$n(\gamma, \alpha_2) = 0$

Lemma 1

If $\gamma : [0, 1] \to C$ *is a closed rectifiable path and* $\alpha \notin \{\gamma\}$, *then*

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-\alpha} \text{ is an integer.}$$

Proof: Define

$$F : [0, 1] \to C \text{ by}$$

$$F(t) = \int_0^t \frac{\gamma'(u)}{\gamma(u)-\alpha}\,du.$$

Hence, $F(0) = 0$, and $F(1) = \int_\gamma \frac{dz}{z-\alpha}$.

Also, $F'(t) = \frac{\gamma'(t)}{\gamma(t)-\alpha}$ $(0 \le t \le 1)$.

We now compute the derivative of the function

$$\begin{aligned}&\frac{d}{dt}(e^{-F(t)}(\gamma(t)-\alpha))\\ &= e^{-F(t)}\gamma'(t) - F'(t)\,e^{-F(t)}(\gamma(t)-\alpha))\\ &= e^{-F(t)}\left[\gamma'(t) - \frac{\gamma'(t)}{\gamma(t)-\alpha}(\gamma(t)-\alpha)\right]\\ &= 0.\end{aligned}$$

Hence, there is a constant function A such that

$$e^{-F(t)}(\gamma(t)-\alpha) = A,$$

so $\gamma(t) - \alpha = Ae^{F(t)}$.

Since γ is a closed path, we have

$$\gamma(0) = \gamma(1)$$

and $Ae^{F(1)} = \gamma(1) - \alpha = \gamma(0) - \alpha = Ae^{F(0)}$.

Since $\gamma(0) - \alpha \neq 0$, we conclude that $A \neq 0$, so that

$$e^{F(0)} = e^{F(1)}.$$

Hence, there is an integer k such that

$$F(1) = F(0) + 2\pi ik.$$

But $F(0) = 0$, so $F(1) = 2\pi ik$ and the lemma is proved.

We now define the index (the winding number) of γ.

Definition: Let γ be a closed path in C. Then for $\alpha \notin \{\gamma\}$

$$n(\gamma, \alpha) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-\alpha}$$

is called **the index (the winding number)** of γ with respect to the point α.

The following properties of the index can be easily checked.

Properties

(*a*) Let γ_1 and γ_2 be closed paths having the same initial points, then

(*i*) $n(\gamma_1, \alpha) = -n(-\gamma_1, \alpha)$ for every $\alpha \notin \{\gamma_1\}$;

(*ii*) $n(\gamma_1 + \gamma_2, \alpha) = n(\gamma_1, \alpha) + n(\gamma_2, \alpha)$ for every

$$\alpha \notin \{\gamma_1\} \cup \{\gamma_2\}.$$

(*b*) Let γ_1 and γ_2 be homotopic closed paths in $C - \{\alpha\}$. Then

$$n(\gamma_1, \alpha) = n(\gamma_2, \alpha) \text{ for every } \alpha \notin \{\gamma_1\} \cup \{\gamma_2\}.$$

Lemma 2

Let γ be a path. Then for $\alpha \notin \{\gamma_1\}$, the function

$$\alpha \to \int_\gamma \frac{dz}{z - \alpha}$$

is a continuous function of α.

Proof: Let $z_0 \notin \{\gamma\}$. We need to prove that

$$\int_\gamma \left(\frac{1}{z - \alpha} - \frac{1}{z - z_0} \right) dz$$

tends to zero as α approaches z_0. Consider the function

$$t \to |\alpha - \gamma(t)|.$$

This function is continuous and not zero. Hence, it has a minimum. Let δ be the minimum distance between the path and the point z_0, *i.e.*

$$\delta = \min_t |z_0 - \gamma(t)|.$$

Now
$$\left| \frac{1}{z - \alpha} - \frac{1}{z - z_0} \right| = \left| \frac{\alpha - z_0}{(z - \alpha)(z - z_0)} \right|.$$

Since $|\alpha - \gamma(t)| \geq \delta/2$ if α is sufficiently close to z_0, we find that

$$\left| \frac{\alpha - z_0}{(z - \alpha)(z - z_0)} \right| \leq \frac{1}{\delta^2/2} |\alpha - z_0|.$$

Hence

$$\left| \int_\gamma \left(\frac{1}{z - \alpha} - \frac{1}{z - z_0} \right) dz \right| \leq \frac{1}{\delta^2/2} |\alpha - z_0| L(\gamma). \tag{1}$$

The right side of (1) tends to zero as $\alpha \to z_0$ and thus the lemma is proved.

Lemma 3

Let γ be a closed path. Let S be a connected set not intersecting γ. Then the function is constant for α in S. If S is unbounded, then this constant is zero.

$$\alpha \to \int_\gamma \frac{dz}{z - \alpha}$$

Proof: We know from Lemma 1 that the integral is the winding number, and is therefore an integer. If a function takes its values in the integers and is continuous, it follows from Lemma 2 that it is constant on any curve. This implies that it is constant on a connected set.

If S is not bounded, then for α arbitrarily large, the integrand has arbitrarily small absolute value, that is

$$\frac{1}{|z-\alpha|} \text{ is arbitrarily small.}$$

An estimate of the integral shows that this constant must be zero.

5.2 STATEMENT OF CAUCHY'S THEOREM

Before stating Cauchy's theorem in a broader set up we need some new definitions.

Definition: Let Ω be an open set. A closed path γ in Ω is **homologous** to 0 in Ω if

$$\int_\gamma \frac{dz}{z-\alpha} = 0$$

for every point α not in Ω.

In other words, $n(\gamma, \alpha) = 0$ for every $\alpha \notin \Omega$.

Let γ_1, γ_2 be closed paths in Ω. γ_1 and γ_2 are said to be homologous in Ω if

$$n(\gamma_1, \alpha) = n(\gamma_2, \alpha) \text{ for every } \alpha \notin \Omega.$$

The following results can be verified easily.

(*i*) Let γ_1 and γ_2 be closed paths in Ω. If γ_1 and γ_2 are homotopic, then they are homologous.

(*ii*) Let γ_1 and γ_2 be closed paths in Ω. If γ_1 and γ_2 are near-together, then they are homologous.

Definition: Let $\gamma_1, \gamma_2, ..., \gamma_n$ be curves and let $m_1, m_2, m_3, ..., m_n$ be integers. A sum of the form

$$\gamma = \sum_{i=1}^{n} m_i \gamma_i$$

is called a **chain.**

We say that γ is a chain in Ω if each γ_i is in Ω. A chain is said to be closed if it is a finite sum of closed paths. We define

$$\int_\gamma f = \sum_{i=1}^{n} m_i \int_{\gamma_i} f$$

where γ is a chain.

If γ is a closed chain where each γ_i is a closed path, then the index of γ with respect to a point α is defined as

$$n(\gamma, \alpha) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-\alpha}$$

where α is not on the chain.

We denote

(*i*) $\gamma_1 \sim \gamma_2$ if $n(\gamma_1, \alpha) = n(\gamma_2, \alpha)$ for every $\alpha \notin \Omega$.

(*ii*) $\gamma_1 \sim 0$, if $n(\gamma_1, \alpha) = 0$ for every $\alpha \notin \Omega$.

Cauchy's Theorem: Let Ω be an open set. Let γ be a closed chain in Ω and let $\gamma \sim 0$ in Ω. Then

$$\int_\gamma f = 0 \text{ for every analytic function in } \Omega.$$

Corollary: Let γ_1 and γ_2 be closed chains in Ω and let $\gamma_1 \sim \gamma_2$ in Ω. Then

$$\int_{\gamma_1} f = \int_{\gamma_2} f.$$

5.3 SOME CONSEQUENCES OF CAUCHY'S THEOREM

We now derive two important consequences of Cauchy's theorem.

Theorem 1. *Let γ be a closed chain in an open set Ω such that $\gamma \sim 0$ in Ω. Let f be analytic on Ω except at a finite number of points $\xi_1, \xi_2, ..., \xi_n$. Let γ_i $(i = 1, 2, ..., n)$ be the boundary of the closed disc $\overline{D_i}(\xi_i)$ contained in Ω. Assume that $\overline{D_i} \cap \overline{D_j} = \phi$ if $i \neq j$.*

Let $m_i = n(\gamma, \xi_i)$

and $\Omega^* = \Omega - \{\xi_1, \xi_2, ..., \xi_n\}$.

Then $\gamma \sim \sum_{i=1}^{n} m_i \gamma_i$ in Ω^*

and $$\int_\gamma f = \sum_{i=1}^{n} m_i \int_{\gamma_i} f.$$

Proof: Let

$$\sigma = \gamma - \sum_{i=1}^{n} m_i \gamma_i$$

and let $\alpha \notin \Omega$. Then

$$n(\sigma, \alpha) = n(\gamma, \alpha) - \sum_{i=1}^{n} m_i n\,(\gamma_i, \alpha)$$
$$= 0.$$

If $\alpha = \xi_k$ for some k, then by Lemma 3

$$n(\gamma_i, \xi_k) = \begin{cases} 1 \text{ if } i = k \\ 0 \text{ if } i \neq k. \end{cases}$$

Hence

$$n(\sigma, \xi_k) = n(\gamma, \xi_k) - m_k = 0.$$

This proves that $\sigma \sim 0$ in Ω^*.

By Cauchy's theorem we conclude that

$$\int_{\gamma} f = \sum_{i=1}^{n} m_i \int_{\gamma_i} f .$$

We illustrate the theorem in Fig. 5.I.

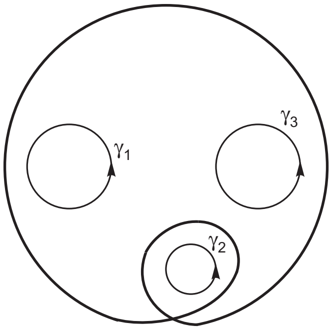


Fig. 5.I

$$\gamma \sim -\gamma_1 - 2\gamma_2 - \gamma_3$$

and $$\int_{\gamma} - f = - \int_{\gamma_1} (-f) - 2\int_{\gamma_2} (-f) - \int_{\gamma_3} (-f) .$$

Theorem 1 will be applied in many cases when Ω is a disc and γ is a circle in Ω. For example, let $\xi_1, \xi_2, \xi_3, ..., \xi_n$ be points inside the circle, as shown in Fig. 5.II. Then

$$\int_{\gamma} f = \sum_{i=1}^{n} \int_{\gamma_i} f$$

where γ_i is small circle around ξ_i.

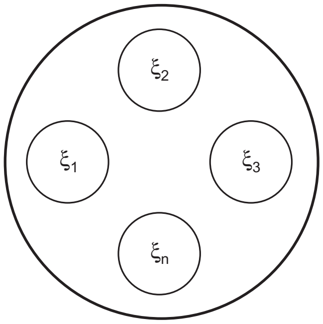


Fig. 5.II

Theorem 2 (Cauchy's Integral Formula). *Let γ be a closed chain in an open set Ω such that $\gamma \sim 0$ in Ω. Let f be analytic on Ω. Then for every z_0 in Ω and not on γ,*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0}\, dz = n(\gamma, z_0)\, f(z_0).$$

Proof: Consider the auxiliary function g defined by

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0}.$$

Note that z_0 is fixed and z is variable. Since $f(z) - f(z_0)$ and $z - z_0$ are both differentiable in Ω, the quotient

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0}$$

is differentiable except at $z = z_0$. By continuity we define

$$g(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}\, f'(z_0).$$

Then g is continuous on Ω and is bounded in a neighbourhood of z_0. Let C_r be the circle of radius r centred at z_0. By Theorem 1 we have

$$(2)\quad \int_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz = \int_\gamma g(z)\, dz = n(\gamma, z_0) \int_{C_r} g(z)\, dz.$$

Since g is bounded in the neighbourhood of z_0 and since the length C_r tends to zero as $r \to 0$, we find that the right side of (2) approaches 0 in absolute value as $r \to 0$. The left side of (2) is independent of r and is therefore equal to 0. Hence

$$\int_\gamma \frac{f(z)}{z - z_0}\, dz = \int_\gamma \frac{f(z_0)}{z - z_0}\, dz = n(\gamma, z_0)\, 2\pi i f(z_0).$$

Thus the theorem is proved.

5.4 APPLICATION OF CAUCHY'S INTEGRAL FORMULA

In the preceding section, we derived Cauchy's integral formula in a broader set up. In this section, we shall apply the formula. In fact, for many applications the simpler form of Cauchy's formula will be used. We restate the theorem. We denote the centre of the disc by α, its radius $\rho > 0$ and assume that f is analytic in a larger disc centred at α.

Theorem 3. *Let f be analytic on the disc $D_r(\alpha)$. Let C_1 be the circle $| z - \alpha | = \rho < r$. Then for any point z_0 in $D_\rho(\alpha)$ we have*

$$(3)\quad f(z_0) = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z - z_0}\, dz.$$

Applying Theorem 3, it will be shown that an analytic function possesses derivatives of all orders. This is a very remarkable result.

Theorem 4. *Let f be analytic in a region Ω. Then $f'(z), f''(z), \ldots, f^n(z), \ldots$ all exist in Ω and are analytic.*

Proof: Let α be a point of Ω. Then it will be proved that f has all derivatives at α.

Let C_1 be a sufficiently small circle with centre α. It follows from (3) that for any point z_0 inside C_1 we have

(4) $f(z_0) = \dfrac{1}{2\pi i}\displaystyle\int_{C_1} \dfrac{f(z)}{z - z_0}\,dz.$

Differentiating (4) n times with respect to z_0, we have

(5) $f^n(z_0) = \dfrac{n!}{2\pi i}\displaystyle\int_{C_1} \dfrac{f(z)}{(z - z_0)^{n+1}}\,dz.$

The validity of (5) will now be established. Choose h sufficiently small so that $z_0 + h$ is in the interior of the disc bounded by C_1. Using (4) we obtain

$$\begin{aligned}\frac{f(z_0+h)-f(z_0)}{h} &= \frac{1}{2\pi i}\frac{1}{h}\int_{C_1} f(z)\left[\frac{1}{z-z_0-h}-\frac{1}{z-z_0}\right]dz\\ &= \frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{(z-z_0-h)(z-z_0)}\,dz\\ &= \frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{(z-z_0)^2}\,dz + J\end{aligned}$$

where

$$J = \frac{h}{2\pi i}\int_{C_1}\frac{f(z)}{(z-z_0)^2\,(z-z_0-h)}\,dz.$$

Since z_0 is inside C_1, $\min|z-z_0|$ for z on C_1 is positive. Denote this minimum by 2δ. It now follows that if $|h| < \delta$ then for z on C_1,

$$|z - z_0 - h| \ge |z - z_0| - |h| > \delta.$$

Since f is uniformly continuous on C_1, there exists some constant M such that

$$|f(z)| \le M \text{ for } z \text{ on } C_1.$$

Let ρ be the radius of C_1.

Then
$$\begin{aligned}|J| &\le \frac{|h|}{2\pi}\frac{M}{(4\delta^2)\delta}2\pi\rho\\ &= \frac{|h|M\rho}{4\delta^3}.\end{aligned}$$

Letting $|h| \to 0$ we get $|J| \to 0$.

This proves the existence of f' and establishes (5) for $n = 1$.

Note that the existence of f' was assured by our assumption that f is analytic. However, repeating the above argument, starting with (5) with $n = 1$, establishes the existence of f'' and proves (5) for $n = 2$. Thus f' has a derivative f'' and so f' is analytic. This proves that if f is analytic then f' is also analytic. Applying this to f' instead of to f we prove that f'' is analytic. More generally, we find that the analyticity of f^n implies that of f^{n+1}. Hence, by induction, it follows that an analytic function has derivatives of all orders which are analytic. We have

$$f^n(z_0) = \frac{1}{2\pi i}\int_{C_1} \frac{f^{(n)}(z)}{z - z_0}\, dz.$$

Integrating by parts n times we obtain (5).

We now establish an important inequality which is called **Cauchy's inequality.**

Theorem 5 (Cauchy' Estimate). *In Theorem 4 assume that there exists a positive constant M such that* $|f(z)| \le M$ *on* $D_r(\alpha)$. *Then*

(6) $|f^n(\alpha)| \le \dfrac{Mn!}{r^n}.$

Proof: In (5), let C_1 be a circle of radius ρ with centre at $z_0 = \alpha$. Then

$$|f^n(\alpha)| = \left| \frac{n!}{2\pi i}\int_{C_1} \frac{f(z)\, dz}{(z-\alpha)^{n+1}} \right|$$

$$\le \frac{n!}{2\pi}\frac{M}{\rho^{n+1}}\int_{C_1} |dz|$$

$$= \frac{n!M}{\rho^n}.$$

Since this holds for all $\rho < r$ and letting $\rho \to r$ we have

$$|f^n(\alpha)| \le \frac{Mn!}{r^n}.$$

From Cauchy's estimate we will prove an important theorem which is known as **Liouville's theorem.** Before starting the theorem we need the definition of entire function.

Definition: A function f is called **entire** if it is analytic on the whole of C.

Theorem 6 (Liouville's Theorem). *A bounded entire function is constant.*

Proof: By hypothesis there is a constant M such that $|f(z)| \le M$. By (6), with $n = 1$,

$$|f'(\alpha)| \le \frac{M}{r} \text{ for arbitrary } r.$$

Hence, $\quad f'(\alpha) = 0.$

Since α is arbitrary, $f'(z) = 0$.

Also, $\quad f(z) - f(0) = \int_0^z f'(\zeta)\, d\zeta.$

It now follows that

$f(z) = f(0)$ which proves the theorem.

We are now in a position to appreciate Liouville's theorem in the following application.

Theorem 7 (Fundamental Theorem of Algebra). *Let f(z) be a non-constant polynomial. Then there is a complex number* z_0 *with*

$$f(z_0) = 0.$$

Proof: Let

$f(z) = a_0 + a_1 z + ... + a_n z^n$ where $a_n \neq 0$. Suppose that $f(z) \neq 0$ for all z. Let

$$g(z) = [f(z)]^{-1}.$$

This function is analytic on the whole of C. Write

$$f(z) = a_n z^n \left(1 + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots + \frac{b_n}{z^n}\right)$$

where $b_1, b_2, b_3, ..., b_n$ are appropriate constants. Observe that $| f(z) |$ is large when $| z |$ is large. Hence $| g(z) | \to 0$ as $| z | \to \infty$. It now follows that there is a number $r > 0$ such that $| g(z) | < 1$ if $| z | > r$. But g is continuous on $\overline{D}(0, r)$, so there is a constant M such that $| g(z) | \leq M$ for $| z | \leq M$. Hence g is a **bounded entire function** and by Liouville's theorem g must be constant. It follows that f must be constant which contradicts our assumption. Hence, the theorem is proved.

Example: Let f be entire function. Suppose that there are constants A and m such that

$$| f(z) | \leq | z |^m \text{ for } | z | \geq A.$$

Then f is a polynomial of degree at most m.

We verify the validity of this example. Write $m \leq k$,

$$\phi(z) = \frac{f(z) - (a_0 + a_1 z + a_2 z^2 + \ldots + a_{k-1} z^{k-1})}{z^k}$$

Observe that $\phi(z)$ is analytic everywhere. Now we have

$$| \phi(z) | \ \leq 1 + \left| \frac{a_0}{z^k} + \frac{a_1}{z^{k-1}} + \ldots + \frac{a_{k-1}}{z} \right| \leq B$$

for some constant B when $| z | \geq A$. When $| z | \leq A$, $| \phi(z) |$ is bounded. Applying Liouville's theorem we find that $\phi(z)$ is a constant. This shows that f is a polynomial of degree at most m.

In Theorem 4, we have proved that an analytic function has derivatives of all orders which are analytic and is represented by the formula (5). Using this result we prove a classical result which is known as **Morera's theorem.**

Theorem 8 (Morera's Theorem). *Let f be continuous in a region Ω and let $\int_\gamma f = 0$ for all closed curves γ in Ω. Then f is analytic in Ω.*

Proof: Let f be analytic in Ω. Define

$$F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta$$

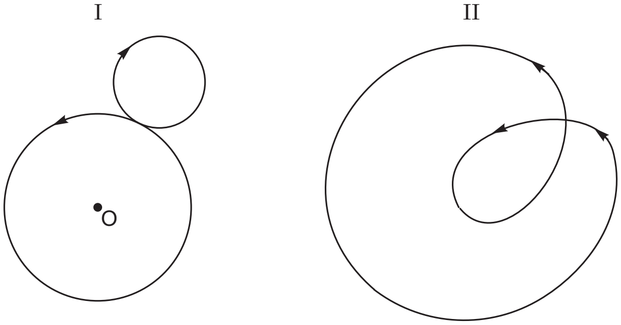
where z is any point in Ω and z_0 is a fixed point in Ω.

The hypothesis implies, as we have already seen in section 4.7, that $f(z)$ is the derivative of an analytic function $F(z)$. It follows from Theorem 4 (section 5.4) that the derivative of an analytic function is itself analytic, that is, *f* is analytic in Ω. This proves Morera's Theorem.

EXERCISES

1. Compute $\int_\gamma \frac{e^z - e^{-z}}{z^4}$

where γ is one of the curves depicted below.



Exercise I

2. Compute $\int_\gamma \left(z + \frac{1}{z}\right) dz$

where γ is the unit circle $\gamma(t) = \text{cis}t$, $0 \le t \le 2\pi$.

3. Using Cauchy's formula for $f^{(n)}(z)$, determine the value of

$$\int_\gamma \frac{P(z)}{(z-c)^k}\, dz$$

where γ is the circle $\gamma(\varphi) = c + r(\cos\varphi + i\sin\varphi)$, and $P(z)$ is a polynomial and k a positive integer.

4. Compute $\int_\gamma \frac{z^2 + 3z + 5}{z+1}\, dz$

where γ is given by $\gamma(t) = a + r(\cos t\ + i \sin t)$, $0 \le t \le 2\pi$, $r > 0$ and $a \in C$.

5. Determine the value of $\int_{c_1} \frac{3z^2 + 7z + 1}{z+1}\, dz$ in the following cases:

(*i*) C_1 is the circle $|z + 1| = 1$.

(*ii*) C_1 is the circle $|z + i| = 1$.

(*iii*) C_1 is the ellipse $x^2 + 2y^2 = 8$.

6. Determine the value of $\int_{c_1} \frac{z+4}{z^2 + 2z + 5}\, dz$ in the following cases:

(*i*) C_1 is the circle $|z| = 1$.

(*ii*) C_1 is the circle $|z + 1 - i| = 2$.

(*iii*) C_1 is the circle $|z + 1 + i| = 2$.

7. Compute $\int_{C_1} \frac{e^z}{(z+1)^2}\, dz$ where C_1 is the circle $|\, z - 1\,| = 3$.

8. Determine the value of $\int_{C_1} \frac{z+1}{z^3 - 2z^2}\, dz$, where

(*i*) C_1 is the circle $|\, z\,| = 1$.
(*ii*) C_1 is the circle $|\, z - 2 - i\,| = 2$.

9. Compute $\int_{\gamma} \frac{2z - i}{z^2 - 2z + 5}\, dz$

in the positive sense around the contour γ of a simply connected domain. Examine all possibilities that can arise in this case.

10. Suppose that f is analytic on $D(0, 1)$ and suppose $|\, f(z)\,| \le 1$ for $|\, z\,| < 1$. Prove that $|\, f'(0)\,| \le 1$.

11. Compute $\int_{\gamma} \left(\frac{z}{z-1}\right)^n dz$

where $\gamma(t) = 1 + e^{it}$, $0 \le t \le 2\pi$ and n is any positive integer.

12. Suppose that f is analytic on the closed unit disk D. Determine the value of the integral

$$\int_D \int f(x + iy)\, dy\, dx.$$

13. Suppose that f is analytic on an open set G. Suppose that $z_0 \in G$ and $f'(z_0) \ne 0$.

Prove that $\frac{2\pi i}{f'(z_0)} = \int_{C_1} \frac{1}{f(z) - f(z_0)}\, dz$ where C_1 is a small circle centred at z_0.

14. Suppose that f is analytic on $D(0, 1)$ and $|\, f(z)\,| \le \frac{1}{(1 - |\, z\,|)}$ for $|\, z\,| < 1$. Determine the best estimate of $|\, f^n(0)|$.

15. Prove that

$$\frac{1}{2\pi i} \int_{\gamma_r} \frac{R + z}{(R - z)z}\, dz = 1$$

where γ_r has the equation $z = re^{i\theta}$, $0 \le \theta \le 2\pi$. Deduce that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos\theta}\, d\theta = 1,\ 0 < r < R.$$

16. Prove that

$$\frac{1}{2\pi i} \int_0^{2\pi} \frac{f(Re^{i\theta})}{Re^{i\theta} - \left(\frac{R^2}{r}\right) e^{i\phi}} (i\, Re^{i\theta}\, d\theta) = 0$$

where C_1 is the circle defined by $z = Re^{i\theta}$ and $z_0 = re^{i\phi}$ for $r < R$.

17. Suppose that f is analytic in the entire (finite) plane and suppose that $f(z)$ is not a constant. Let R and M be any real numbers (no matter how large). Prove that

$$|\, f(z)\,| > M \text{ for } |\, z\,| > R.$$

18. Suppose that $f(z)$ is a polynomial of degree $n > 0$ and let M be any arbitrary positive real number (no matter how large). Prove that $|f(z)| > M$ for $|z| > R$.

19. We recall the Weierstrass' theorem for a real interval $[a, b]$. The theorem states that a continuous function can be uniformly approximated by polynomials.

It is true that every continuous function on the closed unit disk be uniformly approximated by polynomials? Examine this.

20. Recall the definition of the **sup norm,** *i.e.*

$$\|f\| = \sup_{z \in E} |f(z)|$$

where E is any set. We say that $\{f_n\}$ is a **Cauchy sequence** (for the sup norm), if given ε, there exists N such that if $m, n \geq N$, then

$$\|f_n - f_m\| < \varepsilon.$$

Let S be the closure of a bounded open set in C. Let f, g be continuous functions on S. Define their **scalar product**

$$\langle f, g\rangle = \int\int_S f(z)\,\overline{g(z)}\,dy\ dx$$

and define the **associated L_2-norm** by

$$\|f\|_2 = \left\{\int\int_S |f(z)|^2\ dy\ dx\right\}^{1/2}.$$

Prove that $\|f\|_2$ does define a norm.

We define

$$\|f\|_1 = \int\int_S |f(z)|\,dy\ dx.$$

Prove that $f \to \|f\|_1$ is a norm on the space of continuous functions on S. This is called L_1-norm.

(*i*) Suppose that f is analytic on $D(0, r)$ and let $0 < r_1 < r$. Prove that there exists constants $c_1, c_2 > 0$ such that

$$\|f\| \leq c_1\,\|f\|_1 \leq c_2\,\|f\|_2$$

where $\|\ \|$ is the sup-norm on the closed disk of radius r_1 and L_1, L_2 norms also refer to the closed disk of radius r_1.

(*ii*) Suppose that $\{f_n\}$ is a sequence of analytic functions on an open set G, and let $\{f_n\}$ be L_2-Cauchy. Prove that $\{f_n\}$ converges uniformly on compact subsets of G.

6

POWER SERIES

In this chapter, we give an outline of the basic properties of a **power series.** We first recall some elementary facts on infinite series in C.

6.1 INFINITE SERIES IN C

Consider the infinite series

(1) $\sum_{n=0}^{\infty} z_n = z_0 + z_1 + z_2 + \ldots$

where the z_n are arbitrary complex numbers.

Associated with this series is the sequence of its partial sums

(2) $s_n = z_0 + z_1 + \ldots + z_{n-1} + z_n$.

We say that the series $\sum_{n=0}^{\infty} z_n$ converges to z if $\lim_{n \to \infty} s_n$ exists and is equal to z.

In other words, the series $\sum_{n=0}^{\infty} z_n$ converges to z if for every $\varepsilon > 0$ there is an integer n_0 such that

$$\left| \sum_{k=0}^{n} z_k - z \right| < \varepsilon \text{ whenever } n \geq n_0.$$

If this is the case, we say that z is the sum of the infinite series, that is,

$$z = \sum_{k=0}^{\infty} z_k .$$

If $z = \sum_{n=0}^{\infty} z_n$ and $w = \sum_{n=0}^{\infty} w_n$ are two convergent series, with partial sums

$$s_n = \sum_{k=0}^{n} z_k \text{ and } t_n = \sum_{k=0}^{n} w_k ,$$

then the sum converges and

$$z + w = \sum_{n=0}^{\infty} (z_n + w_n).$$

We say that the series $\sum\limits_{n=0}^{\infty} z_n$ converges absolutely if $\sum\limits_{n=0}^{\infty} |z_n|$ converges.

Theorem 1. *If* $\sum\limits_{n=0}^{\infty} z_n$ *converges absolutely, then* $\sum\limits_{n=0}^{\infty} z_n$ *converges.*

Proof: Let $\varepsilon > 0$ and put

$$s_n = \sum_{k=0}^{\infty} z_k .$$

By assumption, $\sum\limits_{n=0}^{\infty} |z_n|$ is convergent. Hence, there is an integer n_0 such that

$$\sum_{n=n_0}^{\infty} |z_n| < \varepsilon$$

Thus if $m > n \geq n_0$,

$$|s_m - s_n| = \left| \sum_{k=n+1}^{m} z_k \right| \leq \sum_{k=n+1}^{m} |z_k|$$

$$\leq \sum_{k=n_0}^{\infty} |z_k| < \varepsilon.$$

That is, $\{s_n\}$ is a Cauchy-sequence and so there is a complex number z with

$$\lim_{n \to \infty} s_n = z.$$

Hence $$\sum_{n=0}^{\infty} z_n = z.$$

If a series does not converge, we say that it diverges or is divergent.

Theorem 2. *Let* $\sum\limits_{n=0}^{\infty} z_n$ *and* $\sum\limits_{n=0}^{\infty} w_n$ *be two absolutely convergent series. Let*

$$c_n = \sum_{k=0}^{n} z_k w_{n-k} = z_0 w_n + z_1 w_{n-1} + \ldots z_n w_0.$$

Then $\sum\limits_{n=0}^{\infty} c_n$ *is absolutely convergent and*

$$(3) \quad \sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} z_n \right) \cdot \left(\sum_{n=0}^{\infty} w_n \right).$$

Proof: We have already seen this theorem during a real analysis course.

Put $$A_n = \sum_{k=0}^{n} z_k, \; B_n = \sum_{k=0}^{n} w_k, \; C_n = \sum_{k=0}^{n} c_k$$

and $$\alpha_n = \sum_{k=0}^{n} |z_k|, \; \beta_n = \sum_{k=0}^{n} |w_k|, \; \gamma_n = \sum_{k=0}^{n} |c_k|.$$

It can be easily checked that γ_n is a monotonically increasing and bounded sequence.

Also,

(4) $|A_{2n} B_{2n} - C_{2n}| \le |\alpha_{2n} \beta_{2n} - \alpha_n \beta_n|$.

It follows from Cauchy-criterion that for sufficiently large n, the right-side of (4) is $\le \varepsilon$.

Hence $$\lim_{n \to \infty} A_{2n} B_{2n} = \lim_{n \to \infty} C_{2n}.$$

That is, $$\left(\sum_{n=0}^{\infty} z_n\right) \cdot \left(\sum_{n=0}^{\infty} w_n\right) = \sum_{n=0}^{\infty} c_n.$$

We now recall the definitions of limit inferior and limit superior of a sequence of real numbers. Let $\{a_n\}$ be a sequence of real numbers. We define

$$\liminf a_n = \underline{\lim}\, a_n = \lim_{n \to \infty} [\inf \{a_n, a_{n+1}, ...\}].$$

$$\limsup a_n = \overline{\lim}\, a_n = \lim_{n \to \infty} [\sup \{a_n, a_{n+1}, ...\}].$$

Note that $\underline{\lim}\, a_n$ and $\overline{\lim}\, a_n$ always exist although they may be $\pm \infty$.

We assume that the reader is familiar with the root and the ratio tests for convergence of infinite series whose terms are real. Here we state the ratio test and the root test for series of complex terms.

Theorem 3 (Ratio Test). *Let* $\sum_{k=0}^{\infty} z_k$ *be a series of non-zero complex terms. Let*

$$\lambda = \lim_{n \to \infty} \inf \left| \frac{z_{n+1}}{z_n} \right|$$

and $$\Lambda = \lim_{n \to \infty} \sup \left| \frac{z_{n+1}}{z_n} \right|.$$

Then

(*i*) *the series* $\sum_{k=0}^{\infty} z_k$ *converges absolutely if* $\Lambda < 1$*;*

(*ii*) *the series* $\sum_{k=0}^{\infty} z_k$ *diverges if* $\lambda > 1$*; and*

(*iii*) *the test gives no information if* $\lambda \le 1 \le \Lambda$.

Theorem 4 (Root Test). *Let* $\sum_{k=0}^{\infty} z_k$ *be a series of complex terms. Let*

$$r = \lim_{n \to \infty} \sup \sqrt[n]{|z_n|}.$$

Then

(*i*) *the series* $\sum_{k=0}^{\infty} z_k$ *converges absolutely if* $r < 1$*;*

(*ii*) *the series* $\sum_{k=0}^{\infty} z_k$ *diverges if* $r > 1$; *and*

(*iii*) *the test gives no information if* $r = 1$.

6.2 SERIES OF FUNCTIONS AND UNIFORM CONVERGENCE

Let E be a set, and f a bounded function on E. Define

$$\| f \| = \sup_{z \in E} | f(z) |.$$

Let $\{f_n\}$, $n = 0, 1, 2, 3, \ldots$, be **a sequence of functions** on E. We say that $\{f_n\}$ converges uniformly on E to a function f if the following property is satisfied.

For every $\varepsilon > 0$ there exists an integer n_0 such that if $n \geq n_0$, then

$$\| f_n - f \| < \varepsilon.$$

We say that $\{f_n\}$ is a Cauchy-sequence for the sup norm, if for every $\varepsilon > 0$, there exists an integer n_0 such that $m, n \geq n_0$, then

$$\| f_n - f_m \| < \varepsilon.$$

Note that for each $z \in E$, the inequality

$$| f_n (z) - f_m (z) | \leq \| f_n - f_m \| \text{ holds.}$$

Hence, for each $z \in E$, the sequence of complex numbers

$$\{f_n (z)\} \text{ converges.}$$

Theorem 5. *Let* $\{f_n\}$ *be a sequence on E which is Cauchy-sequence. Then* $\{f_n\}$ *converges uniformly on E.*

Proof: Suppose that for each $z \in E$,

$$\lim_{n \to \infty} f_n (z) = f(z).$$

Let $\varepsilon > 0$ be given. Then there exists an integer n_0 such that if $m, n \geq n_0$, then

$$| f_n (z) - f_m (z) | < \varepsilon, \text{ for all } z \in E.$$

Let $z \in E$ and let $n \geq n_0$. Choose $m \geq n_0$ sufficiently large (depending on z) such that

$$| f(z) - f_m (z) | < \varepsilon.$$

Then
$$\begin{aligned} | f(z) - f_n (z) | &\leq | f(z) - f_m (z) | + | f_m (z) - f_n (z) | \\ &< \varepsilon + \| f_m - f_n \| \\ &< 2\varepsilon. \end{aligned}$$

This is true for every $n \geq n_0$ and every $z \in E$, which completes the proof.

We say that the series $\sum_{k=0}^{\infty} f_k(z)$ **converges uniformly** on E if the sequence $\{s_n\}$ of **partial sums** defined by

$$s_n (z) = f_0(z) + f_1(z) + \ldots + f_n (z)$$

converges uniformly.

We say that the series $\sum_{k=0}^{\infty} f_k(z)$ **converges absolutely** if

$$\sum_{k=0}^{\infty} | f_k(z) | \text{ converges.}$$

Theorem 6 (Weierstrass-*M*-test). *Let $\{f_n\}$ be a sequence of functions defined on E. Let M_n be a sequence of non-negative real numbers such that*

$$|| f_n || \leq M_n \textit{ for all n.}$$

Then $\sum_{n=0}^{\infty} f_n$ *converges uniformly if* $\sum_{n=0}^{\infty} M_n$ *converges.*

Proof: Let $m \leq n$. Let

$$s_n = \sum_{k=0}^{n} f_k.$$

Then $$|| s_n - s_m || \leq \sum_{k=m+1}^{n} || f_k || \leq \sum_{k=m+1}^{n} M_k.$$

Since $\sum_{n=0}^{\infty} M_n$ converges, it follows from Theorem 5 that

$$\sum_{n=0}^{\infty} f_n \text{ converges uniformly.}$$

Theorem 7. *Let E be a set of complex numbers, and let $\{f_n\}$ be a sequence of continuous functions on E. Suppose that $\{f_n\}$ converges uniformly to f on E.*

Then f is continuous on E.

Proof: The proof is similar to as in the case of the set of real numbers.

Let $z_0 \in E$. Choose n sufficiently large such that

(5) $||f_n - f || < \varepsilon$.

Since $\{f_n\}$ is continuous at z_0, we can choose δ such that whenever

$$| z - z_0 | < \delta \text{ we have}$$

(6) $| f_n (z) - f_n (z_0) | < \varepsilon$

for sufficiently large n as chosen above.

Thus

(7) $| f (z) - f(z_0) | \leq | f (z) - f_n (z) | + | f_n (z) - f_n (z_0) | + | f_n (z_0) - f (z_0) |$.

The first and third terms on the right side of (7) are bounded by

$$|| f - f_n || < \varepsilon.$$

The second term on the right side on (7) is < ε by (6).

Hence

$$| f(z) - f(z_0) | < 3\varepsilon, \text{ and the theorem is proved.}$$

6.3 POWER SERIES

A power series about a is an infinite series of the form

$$(8) \quad \sum_{n=0}^{\infty} a_n (z-a)^n = a_0 + a_1 (z-a) + a_2 (z-a)^2 + \ldots$$

where a and $a_0, a_1, a_2, \ldots$, are complex numbers.

If $a = 0$, we obtain as a particular case a power series

$$(9) \quad \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \ldots .$$

Example (*i*) One of the simplest example of a power series is the geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \ldots .$$

Suppose that a power series (8) is given, we would like to consider all the points z in the complex plane for which the series converges. The following three possibilities arise:

(*i*) The series converges only at $z = a$;

(*ii*) The series converges in the entire complex plane; and

(*iii*) There exists an open disc $D\,(a\,;\,r)$ such that the series converges inside the disc and diverges outside the disc.

The circle $|z-a| = r$ is called the **circle of convergence,** and its radius r is called the **radius of convergence** of the series (8). The radius of convergence of the power series (8) may be determined from the coefficients of the series. In fact, we have the following theorem:

Theorem 8. *With each power series (8) we can associate a real number*

$$(10) \quad r = \left\{\frac{1}{\limsup |a_n|^{1/n}}\right\} \ (0 \le r \le \infty),$$

called the radius of convergence, which has the following properties:

(*i*) *The series converges absolutely in the open disc $D\,(a, r)$;*

(*ii*) *The series diverges outside the closed disc $\overline{D}\,(a, r)$; and*

(*iii*) *The series converges uniformly in every closed disc $\overline{D}\,(a, \rho)$ where $\rho > r$.*

In other words, the series converges uniformly on every compact set $E \subset D$ (a, r).

Proof: Let $$r = \left\{\frac{1}{\limsup |a_n|^{1/n}}\right\} \ (0 \le r \le \infty).$$

We want to show that r is the radius of convergence. If $r = 0$ then no matter how small we choose $\rho = |z-a|$, the expression $|a_n|^{1/n}\rho > k^n$ for an infinite number of terms (k an arbitrary constant). Choosing $k > 1$, we find the sequence $|a_n|\,|z-a|^n$

does not converge to 0 as $n \to \infty$. Hence the series (8) diverges for $| z - a | \neq 0$ and converges to a_0 for $z = a$.

If $r > 0$, *i.e.* if $\limsup | a_n |^{1/n} = 1/r$, then for any ρ, $0 < \rho < r$, choose ρ_0, $0 < \rho < \rho_0 < r$ such that

$$(11) \quad | a_n |^{1/n} < \frac{1}{\rho_0}\left(\frac{1}{\rho_0} > \frac{1}{r}\right)$$

for sufficiently large n.

If follows from (11) that

$$| a_n | \rho^n = | a_n | \rho_0^n \left(\frac{\rho}{\rho_0}\right)^n < \left(\frac{\rho}{\rho_0}\right)^n .$$

Thus the series (8) converges absolutely inside the disc $D(a, r)$.

If $r = \infty$, *i.e.* $\limsup | a_n |^{1/n} = 0$, then however large we choose $\rho = | z - a |$, we have

$$\lim_{n \to \infty} | a_n |^{1/n} \; \rho = 0.$$

In other words, $| a_n | \rho^n < \varepsilon^n$ for sufficiently large n and $0 < \varepsilon < 1$. Comparing the series (8) with the series $\sum_{n=0}^{\infty} \varepsilon^n$, we conclude that the series (8) converges absolutely for all z.

(*ii*) Let $| z - a | = \rho > r$. Then for an infinite number of terms of the series, we have

$$| a_n |^{1/n} > \frac{1}{\rho}$$

i.e.,

$$| a_n | \, | z - a |^n = | a_n | \rho^n > 1.$$

Hence, the series (8) diverges.

(*iii*) Using Weierstrass *M*-Test, we find that the series (8) converges uniformly on every closed disc $\overline{D}(a, \rho)$ where $\rho < r$. Since $E \subset \overline{D}(a, \rho)$ with $\rho < r$, the series (8) converges on any compact set $E \subset D(a, r)$.

When $r = \infty$, the series (8) converges uniformly on any compact subset E of the complex-plane C.

Let Ω be a region and let the open disc $D(a, r) \subset \Omega$. We say that a function f defined in Ω is representable by power series in Ω if to every disc $D(a, r) \subset \Omega$ there corresponds a series (8) which converges to $f(z)$ for all $z \in D(a, r)$.

The geometric series in Example (*i*) represents the function

$$f(z) = \frac{1}{1 - z} \text{ for } | z | < 1.$$

Examples: (*ii*) We define the series

$$\exp(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

It can be easily seen that the radius of convergence of this series is $r = \infty$ and defines a continuous function for all values of z.

(*iii*) The series
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots - = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

and
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

converges at every complex number and the convergence is uniform on each compact subset of C.

These examples give extensions to the exp, sine and cosine functions for the complex numbers.

(*iv*) The series $\sum_{n=0}^{\infty} n!\, z^n = 1 + z + 2z^2 + 6z^3 + \dots$

converges only at $z = 0$.

(*v*) The series $\sum_{n=1}^{\infty} \frac{z^n}{n} = z + \frac{z^2}{2} + \frac{z^3}{3} + \dots$

converges for $|z| < 1$, the series diverges for $|z| > 1$.

At $z = 1$, the series diverges and at $z = -1$, the series converges. This example shows that a series may converge or diverge on the circle of convergence.

(*vi*) The series $1 + \frac{1}{2^2}\left(\frac{z-1}{z+1}\right) + \frac{1}{3^2}\left(\frac{z-1}{z+1}\right)^2 + \frac{1}{4^2}\left(\frac{z-1}{z+1}\right)^3 + \dots$

converges for Re $(z) \geq 0$ and diverges for Re $(z) < 0$.

Using the ratio test, we have
$$\left|\frac{f_{n+1}(z)}{f_n(z)}\right| = \frac{n^2}{(n+1)^2}\left|\frac{z-1}{z+1}\right|.$$

It follows that the series converges absolutely for those values of z for which $\left|\frac{z-1}{z+1}\right| < 1$. Thus the series converges absolutely in the region defined by
$$|z - 1| < |z + 1|.$$

In other words, the series converges absolutely for values of z which lie in the right half of the z-plane.

The series diverges in the region defined by
$$|z - 1| > |z + 1|$$
i.e., for values of z which lie in the left half of the z-plane.

The test fails for values of z which lie on the perpendicular bisector of the line joining $z = 1$ and $z = -1$.

Note that when $\left|\frac{z-1}{z+1}\right| = 1$, the corresponding series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ converges. Hence, the series converges for Re $(z) \geq 0$ and diverges for Re $(z) < 0$.

(*vii*) The series

$$\left(\frac{z}{z+1}\right) + \frac{1}{2^2}\left(\frac{z}{z+1}\right)^2 + \frac{2^3}{3^3}\left(\frac{z}{z+1}\right)^3 + \frac{3^4}{4^4}\left(\frac{z}{z+1}\right)^4 + \ldots$$

converges for Re $(z) > -\frac{1}{2}$ and diverges for Re $(z) \leq -\frac{1}{2}$. Using the root test, we have

$$\sqrt[n]{|f_n(z)|} = \left(1 - \frac{1}{n}\right)\left|\frac{z}{1+z}\right|.$$

It follows that the series converges absolutely for values of z which lie in the region Re $(z) > -\frac{1}{2}$. The series diverges for values of z which lie in the region Re $(z) < -\frac{1}{2}$.

The test fails for values of z for which $\left|\frac{z}{z+1}\right| = 1$.

Note that when $\left|\frac{z}{z+1}\right| = 1$, the corresponding series

$$1 + \frac{1}{2^2} + \frac{2^3}{3^3} + \frac{3^4}{4^4} + \ldots$$

diverges.

(*viii*) The series $1 + z + z^2 + \ldots + z^n + \ldots$

converges uniformly to $\frac{1}{1-z}$ for $0 \leq |z| \leq \rho < 1$, but not for $|z| < 1$.

We have seen in Example (*i*) that this series converges absolutely to $\frac{1}{1-z}$ for $|z| < 1$.

Observe that the remainder

$$R_n(z) = \frac{1}{1-z} - \frac{1-z^{n+1}}{1-z} = \frac{z^{n+1}}{1-z}$$

becomes arbitrary large for real $z = x < 1$ and sufficiently close to 1. Thus for given $\varepsilon > 0$ we cannot find a η_0 which is independent of z and such that $|R_n(z)| < \varepsilon$ for all $n > \eta_0$. Hence, the series is not uniformly convergent in the region $|z| < 1$.

Note that in the region $0 \leq |z| \leq \rho < 1$,

$$\sum_{n=0}^{\infty} |z^n| = \sum_{n=0}^{\infty} |z|^n \leq \sum_{n=0}^{\infty} \rho^n.$$

It follows from Weierstrass *M*-test that the series is uniformly convergent in the region $0 \leq |z| \leq \rho < 1$.

EXERCISES

1. Find the radius of convergence for each of the following series:

(*i*) $\sum_{n=1}^{\infty} n^n z^n$ (*ii*) $\sum_{n=1}^{\infty} \frac{z^n}{n^n}$

(*iii*) $\sum_{n=0}^{\infty} 2^n z^n$ (*iv*) $\sum_{n=1}^{\infty} (\log n)^2 z^n$

(*v*) $\sum_{n=1}^{\infty} 2^{-n} z^n$ (*vi*) $\sum_{n=1}^{\infty} n^2 z^n$

(*vii*) $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$ (*viii*) $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} z^n$

(*ix*) $\sum_{n=0}^{\infty} z^{n!}$ (*x*) $\sum_{n=1}^{\infty} \frac{(z-i)^n}{n^2}$

(*xi*) $\sum_{n=0}^{\infty} 2^{2n}\left(z - \frac{1}{2}\right)^n$ (*xii*) $\sum_{n=0}^{\infty} \pi^n z^{2n}$.

2. Prove that the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$$

is 1, and examine the convergence for $z = 1$, $z = -1$, and $z = i$.

3. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence $r > 0$. Prove that the following series have the same radius of convergence.

(*i*) $\sum_{n=1}^{\infty} n a_n z^n$ (*ii*) $\sum_{n=1}^{\infty} n^2 a_n z^n$

(*iii*) $\sum_{n=1}^{\infty} n^{\alpha} a_n z^n$ for any positive integer α (*iv*) $\sum_{n=1}^{\infty} n a_n z^{n-1}$.

4. Prove that if r is the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$, then the radius of convergence of the series $\sum_{n=0}^{\infty} a_n^2 z^n$ is r^2.

5. Suppose that $\sum_{n=0}^{\infty} a_n$ is a convergent series of complex numbers. Prove that $\sum_{n=0}^{\infty} a_n z^n$ is uniformly convergent on the domain of z such that

$$\pi - \lambda < \arg (z - 1) < \pi + \lambda$$

and $\quad | z - 1 | \le \delta$ where $0 < \lambda < \pi/2$ and $0 < \delta < 2 \cos \lambda$.

6. Suppose that a_n is a decreasing sequence of positive numbers approaching zero. Prove that $\sum_{n=1}^{\infty} a_n z^n$ is uniformly convergent on the domain of z such that $| z | \le 1$ and $| z - 1 | \ge \delta$ where $\delta > 0$.

7. Find the region of convergence and the sum of each of the following series:

(*i*) $1 + (z - i) + (z - i)^2 + (z - i)^3 + \ldots$

(*ii*) $z(1 - z) + z^2 (1 - z) + z^3 (1 - z) + \ldots$

(*iii*) $\frac{1}{2}\frac{z+1}{z-1} + \frac{1}{2^2}\left(\frac{z+1}{z-1}\right)^2 + \frac{1}{2^3}\left(\frac{z+1}{z-1}\right)^3 + \ldots$

(*iv*) $\frac{1}{2(z+i)} + \frac{1}{2^2(z+i)^2} + \frac{1}{2^3(z+i)^3} + \ldots$

8. Determine the region of convergence of each of the following:

(*i*) $1 + \frac{1}{2^2}\frac{\text{Re}\,(z)}{z+1} + \frac{1}{3^2}\left(\frac{\text{Re}\,(z)}{z+1}\right)^2 + \ldots$

(*ii*) $1 + \frac{1}{2^2}\frac{\text{Im}\,(z)}{z+1} + \frac{1}{3^2}\left(\frac{\text{Im}\,(z)}{z+1}\right)^2 + \ldots$

(*iii*) $\frac{4 - z^2}{1^2} + \frac{(4 - z^2)^2}{2^2} + \frac{(4 - z^2)^3}{3^3} + \ldots$

9. Prove that the series

$$z(1 - z) + z^2 (1 - z) + z^3 (1 - z) + \ldots$$

converges uniformly for $| z | \le \rho < 1$ but not for $| z | \le 1$.

10. Find the region of convergence and the sum of the series

$$\frac{z}{(0.z+1)(z+1)} + \frac{z}{(z+1)(2z+1)} + \frac{z}{(2z+1)(3z+1)} + \ldots .$$

Show that the series does not converge uniformly for $| z | \ge \rho > 1$.

11. Find the region of convergence and the sum of the series

$$\frac{1}{z(z+1)} + \frac{1}{(z+1)(z+2)} + \frac{1}{(z+2)(z+3)} + \ldots .$$

Find also the region of uniform convergence of the series.

12. Prove that the Riemann zeta function ζ defined by

$$\zeta (z) = \sum_{n=1}^{\infty} n^{-z}$$

converges for $\text{Re}\,(z) > 1$ and converges uniformly for $\text{Re}\,(z) \ge 1 + \varepsilon$ where $\varepsilon > 0$ is arbitrary small.

7

LAURENT SERIES, SINGULARITIES

7.1 POWER SERIES REPRESENTATION OF ANALYTIC FUNCTIONS

Power series are basic in complex analysis. The following theorem illustrates this fact.

Theorem 1. *Let Ω be a region. If f is represented by the power series in Ω then f is analytic in Ω. Moreover, if*

$$(1)\quad f(z) = \sum_{n=0}^{\infty} a_n \, (z-a)^n$$

for $z \in D_r(a)$, then for these z,

$$(2)\quad f'(z) = \sum_{n=1}^{\infty} n a_n \, (z-a)^{n-1}.$$

Proof: By the root-test we can show that the radius of convergence of both series in (1) and (2) are the same.

Without loss of generality, we can take $a = 0$ and let $g(z)$ be the sum of the series in (2). Fix $w \in D_r(a)$ and choose ρ so that $|w| < \rho < r$. If $z \neq w$, then

$$(3)\quad \frac{f(z)-f(w)}{z-w} - g(w) = \sum_{n=1}^{\infty} a_n \left\{ \frac{z^n - w^n}{z-w} - n w^{n-1} \right\}.$$

If $n = 1$, the expression in brackets in (3) is 0 and if $n \geq 2$, we have the series

$$(4)\quad (z-w) \sum_{k=1}^{n-1} k w^{k-1} z^{n-k-1}.$$

The sum in (4) in absolute value is less than

$$(5)\quad \frac{n(n-1)}{2} \rho^{n-2} \text{ for } |z| < \rho.$$

Thus

$$(6)\quad \left| \frac{f(z)-f(w)}{z-w} - g(w) \right| \leq |z-w| \sum_{n=2}^{\infty} n^2 \left| a_n \right| \rho^{n-2}.$$

Since $\rho < r$, the series in (6) converges. Thus the left side of (6) tends to 0 as $z \to w$. This shows that $f'(w) = g(w)$ and the proof is complete.

Since f' satisfies the same hypothesis as f does, the theorem can be applied to f'. It follows that f has derivatives of all orders, that each derivative is representable by power series in Ω. Thus we obtain

$$(7)\quad f^{(k)}(z) = \sum_{n=k}^{\infty} n\,(n-1)\ldots(n-k+1)\,a_n\,(z-a)^{n-k}$$

if (1) holds.

Hence (1) implies that

$$(8)\quad n!\,a_n = f^{(n)}(a)\ (n = 0, 1, 2, \ldots).$$

7.2 TAYLOR SERIES

In Theorem 1, we have seen that power series with non-zero radius of convergence represent analytic functions. We shall now see that every analytic function can be developed in a convergent Taylor series.

Theorem 2. *Let f be analytic in Ω. Then f can be represented by a power series*

$$f(z) = \sum_{n=0}^{\infty} a_n\,(z-a)^n$$

about each point $a \in \Omega$.

Proof: Let $a \in \Omega$ and let $D(a\,;r)$ be the greatest open disk contained in Ω. Since Ω is open, r is positive. Now choose ρ and ρ_0 such that $0 < \rho < \rho_0 < r$ and denote by γ the boundary of $D(a\,;\rho_0)$, *i.e.*

$$\gamma(t) = a + \rho_0 \cos t,\ 0 \le t \le 2\pi.$$

By Cauchy's formula, we have for each $z \in \overline{D}(a\,;\rho)$

$$(9)\quad f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\,d\xi.$$

Now
$$\frac{1}{\xi - z} = \frac{1}{(\xi - w) - (z - w)} = \frac{1}{\xi - w}\cdot\frac{1}{1 - \dfrac{z-w}{\xi-w}}$$

where
$$|\xi - w| = \rho_0,\ |z - w| \le \rho \text{ and } \left|\frac{z-w}{\xi-w}\right| \le \rho/\rho_0.$$

Then
$$\frac{1}{1 - \dfrac{z-w}{\xi-w}} = 1 + \frac{z-w}{\xi-w} + \left(\frac{z-w}{\xi-w}\right)^2 + \ldots$$

and
$$\frac{1}{\xi - z} = \frac{1}{\xi - w}\left[1 + \frac{z-w}{\xi-w} + \left(\frac{z-w}{\xi-w}\right)^2 + \ldots\right].$$

The geometric series in brackets converges uniformly with respect to ξ on γ and $z \in \overline{D}\,(a\,;\rho)$. Since $f(\xi)$ is continuous function on the compact set γ, $f(\xi)$ is bounded on γ. Therefore

$$(10)\quad \frac{f(\xi)}{\xi - z} = \frac{f(\xi)}{\xi - w} + (z - w)\frac{f(\xi)}{(\xi - w)^2} + \ldots$$

converges uniformly and we can integrate the right side of (10) term by term and obtain

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\,d\xi$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - w}\,d\xi + (z - w)\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - w)^2} + \ldots$$

Setting $$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - w)^{n+1}}\,d\xi \quad (n = 0, 1, 2, \ldots)$$

we find that $f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \ldots$.

Using (8) and noting that

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - w)^{n+1}}\,d\xi = \frac{1}{n!}f^{(n)}(a)$$

we obtain

$$(11)\quad f(z) = \sum_{n=0}^{\infty} \frac{1}{n!}f^{(n)}(a)(z - a)^n$$

$$= f(a) + \frac{(z - a)}{1!}f'(a) + \frac{(z - a)^2}{2!}f^{(2)}(a) + \ldots \frac{(z - a)^n}{n!}f^{(n)}(a) + \ldots$$

The right side of (11) is the Taylor series representation of f about a. This series converges uniformly in $\overline{D}\,(a; \rho)$, $\rho < r$. The reader can verify that this series representation of $f(z)$ is unique.

7.3 ZEROS OF ANALYTIC FUNCTION

Let S be a set of points. Let $a \in S$. We say that a is isolated point of S if there exists a disk $D\,(a\,;r)$ $(r > 0)$ such that $D\,(a\,;r)$ does not contain any point of S other than a. We say that S is discrete if every point of S is isolated.

Let f be analytic near a and let $\sum\limits_{n=0}^{\infty} a_n\,(z - a)^n$ be its Taylor series representation about a. We say that a is a zero of f if $f(a) = 0$. If $f(a) = f'(a) = \ldots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$, then we have $a_0 = a_1 = a_2 = \ldots = a_{n-1} = 0$ and we say that a is a **zero of order *n*** of f. A zero of order one is called a **simple zero.**

If a is zero of order m, then f may be represented as

$$f(z) = \sum_{n=m}^{\infty} a_n (z-a)^n$$

$$= (z-a)^m \sum_{n=m}^{\infty} a_n (z-a)^{n-m}$$

$$= (z-a)^m \sum_{n=0}^{\infty} a_{n+m} (z-a)^n .$$

Denote $h(z)$ by $h(z) = \sum_{n=0}^{\infty} a_{n+m} (z-a)^n .$

Then $h(a) = a_m = \dfrac{f^m(a)}{m!} \neq 0.$

Now h is analytic and therefore continuous. Since $h(a) \neq 0$, $h(z)$ is different from zero near a. The factor $(z-a)^n = 0$ only for $z = a$. Thus $f(z) \neq 0$ near a. We thus obtain that the zeros of finite order of an analytic function are **isolated.** We now state a result and the proof is left to the reader.

Theorem 3. *Let f be analytic of* Ω *and let*

$$f(a) = f'(a) = \ldots f^n(a) = \ldots = 0.$$

Then $f(z) = 0$ *in* Ω.

Example: Define the functions sin z and cos z by

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$$

and

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots$$

The function sin z has simple zeros at $z = 0, \pm \pi, \pm 2\pi, \ldots$. The function $1 - \cos z$ has second order zeros at $z = 0, \pm 2\pi, \pm 4\pi, \ldots$.

7.4 LAURENT SERIES

Buy a Laurent series, we mean a series of the form

$$(12)\quad f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n .$$

Let E be the set of complex numbers. We say that the series (12) converges absolutely (uniformly) on E if the two series

$$(13)\quad f_1(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

and

$$(14)\quad f_2(z) = \sum_{n=1}^{\infty} a_{-n} / (z-a)^n$$

converges absolutely (uniformly) on E. In that case, we write

$$f(z) = f_1(z) + f_2(z).$$

Let $0 \le r_1 < r_2 \le \infty$. We define the **annulus,** say

$$\text{annulus } E = \text{ann}(a\,;\, r_1, r_2) = \{z : r_1 < |z - a| < r_2\}.$$

Notice that ann $(a\,;\,0, r_2)$ is a punctured disk.

In the following theorem, we consider the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \text{ and the annulus } G \text{ defined by}$$

$$\text{ann}(0\,;\, r_1, r_2) = \{z : r_1 \le |z| \le r_2\}.$$

Theorem 4. *Let f be analytic in the annulus G as above. Then f is representable by Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

which converges absolutely and uniformly on $s_1 \le |z| \le s_2$ *where* $r_1 < s_1 < s_2 < r_2$. *Let* γ_1 *and* γ_2 *be the circles of radius* s_1 *and* s_2 *respectively. Then the coefficients* a_n *are given by the formula*

$$(15)\quad a_n = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\xi)\,d\xi}{\xi^{n+1}}, \text{ if } n \ge 0;$$

$$(16)\quad a_n = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\xi)\,d\xi}{\xi^{n+1}}, \text{ if } n < 0.$$

Moreover, the series representation is unique.

Proof: By definition, f is analytic in the annulus

$$r_1 - \varepsilon < |z| < r_2 + \varepsilon.$$

It follows from Lemma 3 of Chapter 5 that the chain $\gamma_2 - \gamma_1$ is homologous to 0. Cauchy's integral formula implies that

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\xi)\,d\xi}{\xi - z} - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\xi)}{\xi - z}\,d\xi$$
$$= f_2(z) + f_1(z).$$

Since f_2 is analytic in the disk $D(0, r_2)$, it has a power series expansion about 0. By Theorem 2, we find that

$$f_2(z) = \sum_{n=0}^{\infty} a_n z^n$$

where the coefficients a_n are given by (15).

Write $\quad \xi - z = -z\left(1 - \dfrac{\xi}{z}\right).$

Then $|\xi/z| \le \dfrac{r_1}{s_1} < 1$, so the geometric series converges and we obtain

$$\frac{1}{z}\cdot\frac{1}{(1-\xi/z)}=\frac{1}{z}\left(1+\frac{\xi}{z}+\left(\frac{\xi}{z}\right)^2+\ldots\right).$$

We can then integrate the series term by term and the desired result follows if we handle them as in the case of the derivation of Cauchy's integral formula.

If the function f is analytic in the annulus ann $(a\,;r_1,r_2)$, then f has a Laurent expansion.

$$(17)\quad f(z)=\sum_{n=-\infty}^{\infty} a_n\,(z-a)^n$$

$$=\sum_{n=0}^{\infty} a_n(z-a)^n+\sum_{n=1}^{\infty} a_{-n}/(z-a)^n\,.$$

The uniqueness of the Laurent series can be verified easily.

The Laurent series is a generalization of the Taylor series. We call $\sum\limits_{n=0}^{\infty}(z-a)^n$ the Taylor part and $\sum\limits_{n=1}^{\infty} a_{-n}/(z-a)^n$ the principal part of the Laurent series. If $a_{-1}=a_{-2}=\ldots=a_{-n}=\ldots=0$, then the principal part vanishes, and the Laurent series reduces to a Taylor series.

Example: We want to find the Laurent series for the function

$$f(z)=\frac{1}{1-z^2}\text{ with centre }z=1.$$

$$\frac{1}{1-z^2}=-\frac{1}{(z-1)(z+1)}.$$

$$\frac{1}{z+1}=\frac{1}{2+(z-1)}=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z-1}{2}\right)^n$$

$$(18)\qquad =\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}\,(z-1)^n.$$

This series converges for $|\,z-1\,|<2$.

Similarly, $\dfrac{1}{z+1}=\dfrac{1}{(z-1)+2}=\dfrac{1}{z-1}\sum\limits_{n=0}^{\infty}\left(-\dfrac{2}{z-1}\right)^n$

$$(19)\qquad =\sum_{n=0}^{\infty}\frac{(-2)^n}{(z-1)^{n+1}}\,.$$

This series converges for $|\,z-1\,|>2$.

Thus, from (18) we obtain

$$f(z)=-\frac{1}{(z-1)(z+1)}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2^{n+1}}\,(z-1)^{n-1}$$

$$= \frac{-\frac{1}{2}}{z-1} + \frac{1}{4} - \frac{1}{8}(z-1) + \frac{1}{16}(z-1)^2 \ldots$$

This series converges in $0 < |z-1| < 2$.

Similarly, from (19) we obtain

$$f(z) = -\frac{1}{(z-1)(z+1)} = -\sum_{n=0}^{\infty} \frac{(-2)^n}{(z-1)^{n+2}}$$

$$= -\frac{1}{(z-1)^2} + \frac{2}{(z-1)^3} - \frac{4}{(z-1)^4} + \ldots$$

This series converges for $|z-1| > 2$.

7.5 ISOLATED SINGULARITIES

Let f be analytic in the annulus ann $(a\,;0, r_2)$, that is, in the punctured disk $D'(a\,;r_2)$. In this case, it may be possible to define $f(a)$ in such a way that f is analytic in the entire disk $D(a\,;r_2)$. In fact, we have the following theorem.

Theorem 5. *Let f be analytic in the punctured disk $D'(a\,;r_2)$. If f is bounded in a neighbourhood of a, then we can define $f(a)$ in a unique way such that the function is also analytic at a.*

Proof: Suppose that $f(z)$ is bounded in a certain neighbourhood of a. Let γ be the circle $|z-a| = \rho$ and let $|f(z)| \le M$ inside and on the boundary of γ.

Write $$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi-a)^{n+1}}\,d\xi, \; (n = 0, \pm 1, \pm 2, \ldots).$$

We see that the Cauchy's inequalities

$$|a_n| \le \frac{M}{\rho^n}$$

is true for negative n, *i.e.*

$$|a_{-n}| \le M\rho^n.$$

Taking ρ arbitrary small, we find that a_{-n} is 0 for $n = 1, 2, 3, \ldots$. Hence, the Laurent series reduces to a Taylor series, and if we define $f(a) = a_0$, f is analytic in the entire disk $D(a\,;r_2)$ (The uniqueness follows from continuity). In this case, we say that a is a **removable singularity of** f.

If f is unbounded near a, then at least one of the coefficients a_{-n} of the principal part of the Laurent series must be different from zero. It may happen that the principal part of the Laurent series (17) has only a finite number of terms,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{m} \frac{a_{-n}}{(z-a)^n}$$

and $a_{-m} \ne 0$. Then f is said to have a **pole** of order m (or **multiplicity** m) at a.

In this case, the function $(z-a)^m f(z)$ is analytic in a neighbourhood of a, and is given by the Taylor series

$$(z-a)^m f(z) = a_{-m} + a_{-m+1}(z-a) + \dots + \dots + a_{-1}(z-a)^{m-1} + a_0 (z-a)^m + \dots$$

If the principal part of the Laurent series has an infinite number of terms, then a is called an **essential singularity.**

Example 1: The function

$$f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2}$$

has a simple pole at $z = 0$ and pole of fifth order at $z = 2$.

Theorem 6. *If f is analytic and has a pole at $z = a$, then $|f(z)| \to \infty$ as $z \to a$ in any manner.*

Proof: Suppose f has a pole of order m at $z = a$. Write

$$f(z) = (z-a)^{-m}\left\{\sum_{n=0}^{\infty} a_n (z-a)^{m+n} + b_m + b_{m-1}(z-a) + \dots + b_1 (z-a)^{m-1}\right\}$$

$$= (z-a)^{-m}\,\psi(z)$$

where ψ is analytic in $|z-a| < r$ and $\psi(a) = b_m \neq 0$.

Hence, $$\lim_{z\to a} |f(z)| = \lim_{z\to a}\left|\frac{\psi(z)}{(z-a)^m}\right| = \infty.$$

7.6 LIMIT POINTS OF ZEROS AND POLES

Observe that the limit point of the sequence of zeros of a non-zero function, analytic in a region Ω, cannot be an interior point of Ω. If it is an interior point of Ω, then either $f(z) = 0$, or else it cannot be analytic at the limit point. It follows that the limit point of the sequence of zeros of a non-zero analytic function is a singularity of $f(z)$. It is an isolated singularity. If an isolated singularity is neither a pole nor a removable singularity, it is called an essential singularity.

Consider the function $f(z) = \sin\frac{1}{z}$. It has zeros at $z = \frac{1}{k\pi}$ $(k = \pm 1, \pm 2, \dots)$. The limit point of these zeros is the point $z = 0$. Hence, $z = 0$ is an isolated essential singularity of $f(z)$.

Let $z_1, z_2, \dots, z_n, \dots$ be a set of points having a limit point z_0 in a region Ω. Let f be analytic in Ω except for poles at $z_1, z_2, \dots$. Then z_0 is also a singularity of $f(z)$. The reason is that f is unbounded in the neighbourhood of z_0. But z_0 is not isolated because it is the limit point of poles. Hence, $f(z)$ has a non-isolated essential singularity at z_0.

Example 2: Consider the function $f(z) = \frac{1}{\cos 1/z}$. This function vanishes for $\frac{1}{z} = \pm(2k+1)\pi/2$, $(k = 0, 1, 2, \dots)$. In other words, this function vanishes for

$z = \pm \dfrac{2}{(2k+1)\pi}$ $(k = 0, 1, 2, ...)$. Hence, $\dfrac{1}{\cos \frac{1}{z}}$ has poles at $z = \pm \dfrac{2}{(2k+1)\pi}$ $(k = 0, 1, 2, ...)$. The point $z = 0$ is the limit point of these poles. Hence, $\dfrac{1}{\cos \frac{1}{z}}$ has a non-isolated essential singularity at $z = 0$.

It can be easily checked that if $f(z)$ has a pole of order m, then $\dfrac{1}{f(z)}$ has a zero of order m at $z = a$, and conversely. The behaviour of a function f near an essential singularity is given by the following theorem.

Theorem 7. *Let f be analytic in the punctured disk $D'(a\,;r_2)$, and let a be an essential singularity of f. Then the image of an arbitrarily small disk $D(a\,;\varepsilon)$ is everywhere dense in the complex plane. In other words, the values of f on $D(a\,;\varepsilon)$ come arbitrarily close to any complex number.*

Proof: Suppose the theorem is false. There exists a complex number c and a positive number ρ such that $|f(z) - c| > \rho$ for all $z \varepsilon D'(z\,;\varepsilon)$. Consider the function

$$g(z) = \frac{1}{f(z) - c}.$$

g is analytic and bounded in $D'(a\,;\varepsilon)$. Hence a is a removable singularity of g, and g possesses an analytic extension for the entire disk $D(a\,;\varepsilon)$. It, then, follows that $\dfrac{1}{g(z)}$ has a pole at a, which means that $f(z) - c$ has a pole, contradicting the hypothesis that $f(z)$ has an essential singularity. This completes the proof.

Picard has proved that, in an arbitrarily small neighbourhood of an essential singularity, f not only comes arbitrarily close to every complex number, but takes on every complex value except possibly one. For instance, the function $e^{1/z}$ omits the value 0, so it is necessary to allow for this one omission. The proof of Picard's Theorem is beyond the scope of this book.

7.7 MEROMORPHIC FUNCTIONS

We say that f is **meromorphic** in the region Ω if the only singularities of f in Ω are poles.

Examples: (*i*) $\dfrac{e^z}{z}$ is meromorphic in the entire plane.

(*ii*) A rational function is meromorphic in the entire plane.

(*iii*) The function $\dfrac{1}{\sin z}$ is meromorphic in the entire plane. The only singularities are simple poles at $z = k\pi$, $(k = 0, \pm 1, \pm 2, ...)$.

Riemann Sphere: In complex analysis, we will be concerned with functions that become infinite as the variable approaches a given point. To discuss this situation we introduce the **extended-plane** $C_\infty \equiv C \cup \{\infty\}$. Let f be a function defined on C_∞. Let $t = \frac{1}{z}$, and define

$$\varphi(t) = f\left(\frac{1}{t}\right) \text{ for } t \neq 0, \infty.$$

We say that f has a removable singularity, a pole, or an essential singularity at infinity if φ has, respectively, a removable singularity, a pole, or an essential singularity at $z = 0$. If f has a pole at ∞ then the order of the pole is the order of the pole of φ at $z = 0$.

We say that f is meromorphic on C_∞ if f is meromorphic on C and is also meromorphic at infinity. We say that f is analytic on C_∞ if f is analytic on C and is also analytic at infinity.

Examples: (*i*) The function e^z has an essential singularity at infinity.

(*ii*) The polynomial $P(z) = \sum\limits_{k=0}^{n} a_k z^k$ has a pole of order n at infinity.

(*iii*) The rational function $P(z)/Q(z)$ has a pole of order $p - q$ at infinity if the degree p of $P(z)$ is greater than the degree q of $Q(z)$. The rational functions are meromorphic on the Riemann sphere S.

Example: Find the Laurent series for the function $e^{1/z}$ in $0 < |z| \leq \infty$. Using this expansion show that for $n = 0, 1, 2, 3, \ldots$

$$\frac{1}{\pi}\int_0^x \exp(\cos\theta)\cos(\sin\theta - n\theta)\, d\theta = \frac{1}{n!}.$$

Note that $1/z$ is analytic for $|z| > 0$ and has a removable singularity at ∞. It follows that $e^{1/z}$ has removable singularity at ∞. Hence

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \ldots + \frac{1}{k!\,z^k} + \ldots .$$

The coefficient of z^k in this series is given by

$$a_k = \frac{1}{2\pi i}\int_\gamma e^{1/z} z^{-k-1}\, dz$$

where γ is any circle $|z| = r > 0$.

Take $r = 1$ and $z = e^{i\theta}$. Then

$$e^{1/z} = \exp(\cos\theta - i\sin\theta) = \exp(\cos\theta)\cdot\exp(-i\sin\theta).$$

The integral becomes

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(\cos\theta)\exp(-i(\sin\theta + k\theta))\, d\theta.$$

Since the integrand is an odd function, it follows that

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(\cos\theta)\cos(\sin\theta + k\theta)\,d\theta.$$

Note that the coefficient a_k can be read off by inspection and setting $k = -n$ we find that

$$\frac{1}{\pi}\int_{0}^{\pi} \exp(\cos\theta)\cos(\sin\theta - n\theta)\,d\theta = \frac{1}{n!}.$$

EXERCISES

1. Expand the following functions in a Taylor series about the point $z = 1$:

(*i*) $f(z) = \dfrac{1}{z+2}$ (*ii*) $f(z) = \log z$.

2. Find two distinct Laurent expansions for $f(z) = \dfrac{(3z+1)}{z^2-1}$ around $z = 1$. Examine the convergence of each series.

3. Give two different Laurent expansions for

$$f(z) = \frac{1}{z^2(z-i)}$$

around $z = i$. Examine the convergence of each series.

4. Give the Laurent expansion of $f(z) = \dfrac{1}{z(z-1)(z-2)}$ in each of the following annuli

(*i*) ann (0, 0, 1), (*ii*) ann (0, 1, 2), (*iii*) ann (0, 2, ∞).

5. Show that the coefficients in the Laurent expansion of $f(z) = \sin\left[z + \left(\dfrac{1}{z}\right)\right]$ in powers of z are given by the formula

$$a_n = \frac{1}{2\pi}\int_{0}^{2\pi} \cos n\theta \sin(2\cos\theta)\,d\theta.$$

6. Determine the zeros of the following functions and their order

(*i*) $e^z - 1$, (*ii*) $\sin^2 z$, (*iii*) $\sin z^2$, (*iv*) $\dfrac{\sin z}{z}$.

7. Prove that $f(z) = \tan z$ is analytic is C except for simple poles at $z = \pi/2 + n\pi$, for each integer n. Determine the singular part of f at each of these poles.

8. Suppose that $f: G \to C$ is analytic except for poles. Show that the poles of f cannot have a limit point in G.

9. Expand in a Laurent series the following functions:

(*i*) $f(z) = \dfrac{1-e^{2z}}{z^4}$ about the origin.

(*ii*) $f(z) = \dfrac{1}{z^2\,(z-1)}$ about $z = 0$ and $z = 1$.

(*iii*) $f(z) = \dfrac{e^{2z}}{(z-1)^2}$ about $z = 1$.

(*iv*) $f(z) = \dfrac{1+e^z}{\sin z + z\cos z}$ about the origin.

10. Show that the singularities in each of the following functions is a pole. Find the location and order of each pole

(*i*) $(e^z - 1)^{-z}$, (*ii*) $z^{-3}\exp(z^2)/2$, (*iii*) $z^2/(z+1)^2\,(z-2)$.

11. Determine the singularities in each of the following functions. Find the singular part if it is a pole. Define $f(0)$ if it is a removable singularity so that f is analytic at $z = 0$.

(*i*) $\dfrac{\log(z+1)}{z^2}$ (*ii*) $\dfrac{z^2+1}{z(z-1)}$ (*iii*) $\dfrac{4}{1-e^z}$

(*iv*) $z^n \sin\dfrac{1}{z}$ (*v*) $z\cos\dfrac{1}{z}$.

12. Show that an entire function has a removable singularity at infinity iff it is a constant.

13. Show that an entire function has a pole at infinity of order m iff it is a polynomial of degree m.

14. Show that the image of an entire function is dense in C.

15. **Isomorphism:** Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let $f : X_1 \to X_2$ be a function. A one-to-one correspondence f is called a homeomorphism if f and f^{-1} are both continuous. Let Ω be a region. Let f be complex differentiable on Ω. Let f be a homeomorphism $\Omega \to f(\Omega)$. It can be easily seen that f^{-1} is also complex differentiable on $f(\Omega)$.

We call a **complex differentiable homeomorphism** an **isomorphism.**

If f is an isomorphism $\Omega \to f(\Omega)$, then Ω is said to be **isomorphic** to $f(\Omega)$. If $f(\Omega) = \Omega$, then the isomorphism f is called an **automorphism** of Ω.

Prove that the only **analytic automorphism** of C are the funcitons of the form $f(z) = cz + d$, where c, d are constants and $c \neq 0$.

16. Let U and V be open sets. Suppose that f is meromorphic on U and suppose that $\varphi : V \to U$ is an analytic isomorphism. Let $\phi(z_0) = w_0$, and let f has order n at w_0. Prove that f o φ has order n at z_0.

In other words, the order is invariant under analytic isomorphisms.

Note that n is a positive or negative integer.

8

RESIDUE THEOREM AND ITS APPLICATIONS

8.1 DEFINITION

Let f has an isolated singularity at $z = z_0$ and let

(1) $$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

be its Laurent expansion. We call a_{-1} the **residue** of f at z_0, and write

$$a_{-1} = \text{Res}\,(f, z_0).$$

Examples: (*i*) Res $(e^{1/z}, 0) = 1$.

(*ii*) $\text{Res}\left(\cos\left(\frac{1}{z}\right), 0\right) = 0.$

We give below some methods for calculating the residue of a function at a pole.

If $f(z)$ has a simple pole at a point $z = z_0$, then

$$f(z) = \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $a_{-1} \neq 0$.

$$\text{Res}\,(f, z_0) = a_{-1} = \lim_{z \to z_0} (z - z_0) f(z).$$

Example: Let

$$f(z) = \frac{e^{iz}}{z^2 + a^2}, \ (a \text{ real})$$

The function has simple poles at $z = ai$ and $z = -ai$.

$$\text{Res}\,(f, ai) = \lim_{z \to ai} (z - ai) \frac{e^{iz}}{z^2 + a^2}$$

$$= e^{-a}/2ai.$$

Similarly,

$$\text{Res}\,(f, -ai) = -e^{a}/2ai.$$

In the case of a simple pole, another formula is obtained as follows:

If $f(z)$ has a simple pole at $z = z_0$, we write

$$f(z) = g(z) / h(z)$$

where h has a simple zero at z_0 and $g(z_0) \neq 0$.

$$\begin{aligned}\operatorname{Res}(f, z_0) &= \operatorname{Res}\left(\frac{g}{h}, z_0\right)\\ &= \lim_{z \to z_0} (z - z_0)\frac{g(z)}{h(z)}\\ &= \lim_{z \to z_0} \frac{g(z)}{\dfrac{h(z) - h(z_0)}{z - z_0}} \quad (h(z_0) = 0)\\ &= g(z_0) / h'(z_0).\end{aligned}$$

Example: Let

$$f(z) = \frac{7 - 3z}{z^2 - z}.$$

The function has simple poles at $z = 0$ and $z = 1$.

$$\operatorname{Res}(f, 0) = \left(\frac{7 - 3z}{2z - 1}\right)_{z = 0} = -7.$$

$$\operatorname{Res}(f, 1) = \left(\frac{7 - 3z}{2z - 1}\right)_{z = 1} = 4.$$

If $f(z)$ has a pole of order $m > 1$ at a point $z = z_0$, then

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \ldots + \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n. \tag{1}$$

Multiplying both sides of (1) by $(z - z_0)^m$ we get

$$\begin{aligned}&(z - z_0)^m f(z) = g(z)\\ &= a_{-m} + \ldots + a_{-1}(z - z_0)^{m-1} + a_0 (z - z_0)^m + \ldots + a_n (z - z_0)^{m+n} + \ldots .\end{aligned}$$

This shows that the residue of $f(z)$ at $z = z_0$ is the coefficient of $(z - z_0)^{m-1}$ in the Taylor expansion of

$$g(z) = (z - z_0)^m f(z).$$

It follows from Taylor's Theorem that

$$a_{-1} = \frac{g^{(m-1)}(z_0)}{(m - 1)!}.$$

Hence, we have the following result: If f has a pole of order m at $z = z_0$ and put $g(z) = (z - z_0)^m f(z)$, then

$$\operatorname{Res}(f, z_0) = \frac{g^{(m-1)}(z_0)}{(m - 1)!}.$$

Example: Let

$$f(z) = \frac{2z}{(z+4)(z-1)^2}$$

The function has a pole of order two at $z = 1$.

$$\text{Res}(f, 1) = \lim_{z \to 1} \frac{d}{dz}((z-1)^2 f(z)) = \frac{8}{25}.$$

Theorem 1. *Let f has an isolated singularity at $z = z_0$. Let γ be a small circle (centred at z_0) such that f is analytic on γ and its interior, except the centre z_0. Then*

$$\text{Res}(f, z_0) = \frac{1}{2\pi i} \int_\gamma f(z)\, dz.$$

Proof: Since $f(z)$ has an isolated singularity at $z = z_0$, then we can represent it by its Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}$$

valid throughout the circle γ except $z = z_0$ itself. This series converges uniformly and absolutely for z on the circle, we can integrate it term by term. The integral of $(z-z_0)^n$ over the circle is equal to 0 for all values of n except $n = -1$. In this case, the value of the integral is equal to $2\pi i$ (cf. Example 1, § 4.3). This completes the proof.

Inserting Theorem 1 in Theorem 1 of Chapter 5 we obtain the generalized Residue Theorem.

Theorem 2 (Residue Theorem). *Let Ω be a region. Let γ be a closed chain in Ω such that γ is homologous to 0 in Ω. If f is analytic in Ω except at a finite number of points $z_1, z_2, \ldots z_n$, then*

$$\int_\gamma f = \sum_{i=1}^{n} m_i \,\text{Res}(f, z_i)$$

where $m_i = \eta(\gamma, z_i)$.

Theorem 2 is very important and used quite often when Ω is simply connected region. In applications, Ω will be a disk, or inside of a rectangle, where the simply connectedness is obvious.

8.2 APPLICATIONS OF THE RESIDUE THEOREM

This section is devoted to calculating certain integrals by means of Residue Theorem.

(*i*) We consider integrals of the type

$$(2) \quad I = \int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$$

where $R(\cos\theta, \sin\theta)$ is a real rational function of $\cos\theta$ and $\sin\theta$. Put $z = e^{i\theta}$,, $(0 \le \theta \le 2\pi)$. Then

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right);$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

We find that the integrand becomes a rational function of z. Since $d\theta = dz/iz$, the given integral takes the form

$$(3) \quad I = \int_{|z|=1} f(z)\,\frac{dz}{iz}.$$

Example:

$$I = \int_0^{2\pi} \frac{d\theta}{1 - 2p\cos\theta + p^2} \quad (0 < p < 1).$$

Setting $z = e^{i\theta}$, we find

$$I = \int_{|z|=1} \frac{dz/iz}{1 - 2p\,\frac{1}{2}\left(z + \frac{1}{z}\right) + p^2}$$

$$= \int_{|z|=1} \frac{dz}{i(1-pz)(z-p)}.$$

The singularities of $\frac{1}{(1-pz)(z-p)}$ are simple poles at $z = \frac{1}{p} > 1$ and $z = p < 1$. The only singularity inside the unit circle is a simple pole at $z = p < 1$.

$$\text{Res}(f, p) = \lim_{z \to p} (z-p) f(z)$$

$$= \lim_{z \to p} \frac{1}{i(1-pz)}$$

$$= \frac{1}{i(1-p^2)}.$$

Thus by Residue theorem

$$\int_0^{2\pi} \frac{d\theta}{1 - 2p\cos\theta + p^2} = 2\pi i\,\frac{1}{i(1-p^2)}$$

$$= \frac{2\pi}{1-p^2} \quad (0 < p < 1).$$

(*ii*) Consider the integrals of the type

$$(4) \qquad I = \int_{-\infty}^{\infty} f(x)\,dx$$

where $f(z) = \frac{g(z)}{h(z)}$ is a rational function with no poles on the real axis. The degree of $h(z)$ is at least two units higher than the degree of $g(z)$. We consider the corresponding complex integral

$$\int_\gamma f(z)\,dz$$

around a path γ as shown in Fig. 8.I.

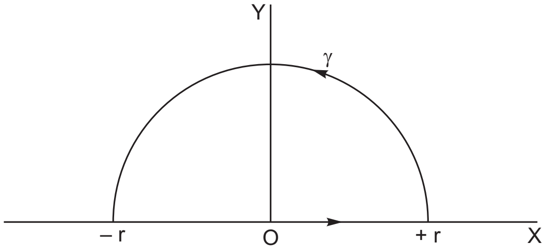


Fig. 8.I

The boundary γ consists of the segment $(-r, r)$ on the real axis and of the semi-circle s centred at the origin. If γ is large enough, all the poles of $f(z)$ in the upper half plane will lie inside γ, and we have

$$\int_\gamma f(z)\,dz = \int_{-r}^{r} f(x)\,dx + \int_s f(z)\,dz$$
$$= 2\pi i \sum_{\operatorname{Im} z_j > 0} \operatorname{Res}(f, z_j)$$

where z_j, $j = 1, 2, \ldots$ are the poles of $f(z)$.

Let $r \to \infty$, then

$$\lim_{r\to\infty} \int_{-r}^{r} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

(Since I is a convergent integral,

$$\lim_{r\to\infty} \int_{-r}^{r} f(x)\,dx = \lim_{\substack{a\to\infty \\ b\to\infty}} \int_{-a}^{b} f(x)\,dx).$$

Since the degree of $h(z)$ is at least two units higher than the degree of $g(z)$, there exists a constant M such that

$$|f(z)| < \frac{M}{r^2}$$

for sufficiently large r and for all $z \in s$. It follows that

$$\left| \int_s f(z)\,dz \right| \le \pi r \frac{M}{r^2} = \frac{\pi M}{r}.$$

Hence

$$\lim_{r\to\infty} \int_s f(z)\,dz = 0$$

and
$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{\operatorname{Im} z_j > 0} \operatorname{Res}(f, z_j).$$

Example: $$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$

The function $f(z) = \dfrac{1}{1+z^4}$ has four simple poles at the points

$$z_1 = e^{\pi i/4},\ z_2 = e^{3\pi i/4},\ z_3 = e^{-3\pi i/4},\ z_4 = e^{-\pi i/4}.$$

The first-two poles lie in the upper half plane

$$\text{Res}\,(f, z_1) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_1}$$

$$= \frac{1}{4}e^{-3\pi i/4} = -\frac{1}{4}e^{\pi i/4}.$$

Similarly $\quad \text{Res}\,(f, z_2) = \dfrac{1}{4}e^{-\pi i/4}.$

Thus $\quad I = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = 2\pi i\left(\frac{-e^{\pi i/4} + e^{-\pi i/4}}{4}\right)$

$$= \frac{\pi}{\sqrt{2}}\,.$$

(*iii*) Consider integrals of the type

(5) $\quad I = \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{ix}\, dx$

where f is real valued on the real axis. The real and imaginary parts of this integral are

$$\int_{-\infty}^{\infty} f(x) \cos dx \text{ and } \int_{-\infty}^{\infty} f(x) \sin x\, dx$$

respectively.

We assume that f is analytic in the upper half plane except at $z_1, z_2, ..., z_n$ which do not lie on the real axis. We also assume that $\lim\limits_{z\to\infty} f(z) = 0$ in the upper half plane.

Now take the rectangle γ with vertices B, $B + iC$, $-A + iC$, $-A$ (Fig. 8.II). Choose A, B and C sufficiently large to ensure that all the singular points $z_1, z_2, ..., z_n$ lie inside the rectangle. Then

$$\int_\gamma f(z)\, e^{iz}\, dz = 2\pi i \sum_{j=1}^{n} \text{Res}\,(f(z)\, e^{iz}, z_j).$$

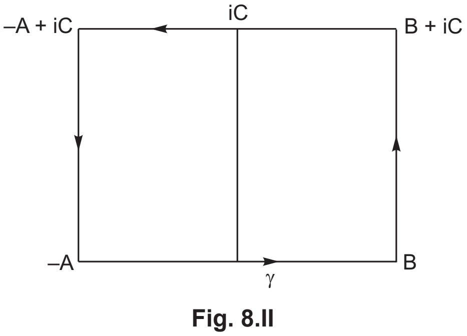


Fig. 8.II

It follows that

$$(6)\quad \int_{-A}^{B} f(x)\, e^{ix}\, dx = 2\pi i \sum_{j=1}^{n} \operatorname{Res}\,(f(z)\, e^{iz}, z_j)$$
$$- \int_{B}^{B+iC} f(z)\, e^{iz}\, dz + \int_{-A+iC}^{B+iC} f(z)\, e^{iz}\, dz + \int_{-A}^{-A+iC} f(z)\, e^{iz}\, dz.$$

The three integrals in the right side of (6) converges to 0 as A, $B \to \infty$ independently and $C \to \infty$.

For,

$$I_1 = \int_{B}^{B+iC} f(z)\, e^{iz}\, dz.$$

Put $z = B + iy$, then

$$I_1 = \int_{0}^{C} f(B+iy)\, e^{i(B+iy)}\, i\, dy.$$

Hence $$|\, I_1 \,| \le M(B) \int_{0}^{C} e^{-y}\, dy \le M(B).$$

where $M(B) = \max\limits_{0 \le y \le C} |f(B+iy)|$.

Similarly $$|\, I_3 \,| = \left| \int_{-A}^{-A+iC} f(z)\, e^{iz}\, dz \right| \le M(A)$$

where $M(A) = \max\limits_{0 \le y \le C} |f(-A+iy)|$.

Let $$I_2 = \int_{-A+iC}^{B+iC} f(z)\, e^{iz}\, dz.$$

Put $z = x + iC$, then

$$|\, I_2 \,| = \left| \int_{-A}^{B} f(x+Ci)\, e^{i(x+iC)}\, dx \right|$$
$$\le M(C) \int_{-A}^{B} e^{-C}\, dx$$
$$= M(C)\, e^{-C}\,(A+B).$$

Since $\lim\limits_{z \to \infty} f(z) = 0$ in the upper half plane, we can choose A and B sufficiently large such that $M(A) \le \varepsilon$, $M(B) \le \varepsilon$ and C sufficiently large such that

$$M(C)\, e^{-C}\,(A+B) < \varepsilon.$$

It follows that $|\, I_1 \,| + |\, I_2 \,| + |\, I_3 \,| < 3\varepsilon$. Thus we have

$$\int_{-\infty}^{\infty} f(x)\, e^{ix}\, dx = 2\pi i \sum_{j=1}^{n} \operatorname{Res}\,(f(z)\, e^{iz}, z_j).$$

Example:

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx.$$

In fact, $\dfrac{e^{iz}}{1+z^2}$ has only one pole in the upper half plane *i.e.* a simple pole at $z = i$.

$$\text{Res}\left(\frac{e^{iz}}{1+z^2}, i\right) = \frac{1}{2ie}.$$

Therefore

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = 2\pi i\,\frac{1}{2ie} = \frac{\pi}{e}$$

and this gives the result.

We have assumed that the function has no singularities on the real axis. If f has singular points on the real axis, say a simple pole at $z = z_1$, then we modify the path γ by replacing the interval $[z_1 - \varepsilon, z_1 + \varepsilon]$ on the real axis by the semi-circle $S_\varepsilon(z_1)$ as in the Fig. 8. III.

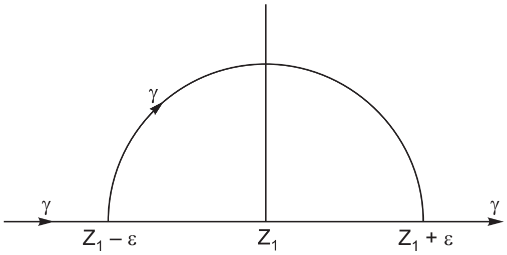


Fig. 8.III

Note that if $f(z)$ has a simple pole at z_1, then $f(z)\,e^{iz}$ has also a simple pole at z_1. Let

$$f(z)\,e^{iz} = g(z) + \frac{a_{-1}}{z - z_1}$$

where $g(z)$ is the Taylor part.

Then

$$\int_{S_\varepsilon} f(z)\,e^{iz}\,dz = \int_{S_\varepsilon} g(z)\,dz + \int_{S_\varepsilon} \frac{a_{-1}}{z - z_i}\,dz.$$

Now $$\left|\int_{S_\varepsilon} g(z)\,dz\right| \le M(\varepsilon)\,\pi\varepsilon$$

where $M(\varepsilon) = \max\{|g(z)| : z \in S_\varepsilon\}$.

Since $g(z)$ is continuous at z_1, it follows that

$$\lim_{\varepsilon\to 0}\int_{S_\varepsilon} g(z)\,dz = 0$$

and $$\int_{S_\varepsilon} \frac{a_{-1}}{z - z_1}\,dz = a_{-1}\,\pi i.$$

Hence, $\lim\limits_{\varepsilon \to 0} \int_{S_\varepsilon} f(z)\, e^{iz}\, dz = a_{-1}\, \pi i.$

Thus,
$$\lim_{\varepsilon \to 0}\left(\int_{-\infty}^{z_1 - \varepsilon} f(x)\, e^{ix}\, dx + \int_{z_1+\varepsilon}^{\infty} f(x)\, e^{ix}\, dx\right)$$
$$= 2\pi i\left[\sum_{\operatorname{Im} z_j > 0} \operatorname{Res}(f(z)\, e^{iz}, z_j) + \frac{a_{-1}}{2}\right].$$

Example:
$$I = \int_0^\infty \frac{\sin x}{x}\, dx.$$

Since $\dfrac{\sin x}{x}$ is an even function, we have
$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \frac{1}{2}\operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx\right).$$

Consider the function $\dfrac{e^{iz}}{z}$. This function has no singular points in the upper half plane, therefore
$$\lim_{\varepsilon \to 0}\left(\int_{-\infty}^{\varepsilon} \frac{\sin x}{x}\, dx + \int_{\varepsilon}^{\infty} \frac{\sin x}{x}\, dx\right)$$
$$= \frac{1}{2}\operatorname{Im}(\pi i a_{-1}).$$
$$a_{-1} = \operatorname{Res}\left(\frac{e^{iz}}{z}, 0\right) = 1.$$

Hence
$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \frac{\sin x\, dx}{x} = \pi / 2.$$

Note that $\displaystyle\int_{\varepsilon}^{\infty} \frac{\sin x}{x}\, dx$ is a continuous function of ε, hence
$$\int_0^\infty \frac{\sin x}{x}\, dx = \pi / 2.$$

(*iv*) Consider the integrals of the type

(7) $I = \displaystyle\int_0^\infty x^\alpha f(x)\, dx \; (0 < \alpha < 1)$

where $f(z) = \dfrac{g(z)}{h(z)}$ is a rational function analytic on C, except for a finite number of poles, none of which lies on the positive real axis $x > 0$.

We assume that the degree of $h(z)$ is greater than the degree of $g(z)$ by at least two units.

Suppose α is real positive number. We define $z^\alpha = e^{\alpha \log z}$ where the log is defined on the simply connected set equal to the plane from which the axis $x \geq 0$ has been deleted.

In order to evaluate the integral, we choose the closed path γ as in Fig. 8.IV. Then γ consists of the segment $[\varepsilon, r]$ and the circle $S_r(0)$ (taken counterclockwise), the segment $[r, \varepsilon]$ and the circle $S_\varepsilon(0)$ (taken clockwise). Let $z_1, z_2, ..., z_n$ be the poles of f inside γ. Then

$$\int_\gamma z^\alpha f(z)\, dz = 2\pi i \sum_{j=1}^{n} \text{Res}\,[z^\alpha f(z), z_j].$$

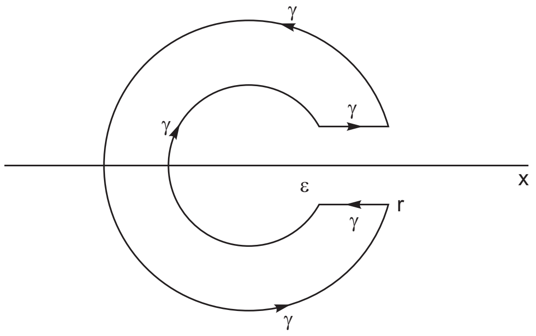


Fig. 8.IV

We suppose that ε is sufficiently small and r sufficiently large so that all the singular points of f except the origin lie inside γ.

Now

$$\int_\gamma z^\alpha f(z)\, dx = \int_\varepsilon^r x^\alpha f(x)\, dx + \int_{S_r} z^\alpha f(z)\, dz$$

$$+ \int_s^\varepsilon x^\alpha e^{2\pi i\alpha} f(x)\, dx - \int_{S_\varepsilon(0)} z^\alpha f(z)\, dz.$$

$$= 2\pi i \sum_{j=1}^{n} \text{Res}\,[z^\alpha f(z), z_j]$$

or

$$(1 - e^{2\pi i\alpha}) \int_\varepsilon^r x^\alpha f(x)\, dx$$

$$(8) \quad = 2\pi i \sum_{j=1}^{n} \text{Res}\,(z^\alpha f(z), z_j) - \int_{S_r} z^\alpha f(z)\, dz + \int_{S_\varepsilon(0)} z^\alpha f(z)\, dz.$$

As $\varepsilon \to 0$ and $r \to \infty$, we have

$$\lim_{\substack{\varepsilon \to 0 \\ r \to \infty}} \int_\varepsilon^r x^\alpha f(x)\, dx = \int_0^\infty x^\alpha f(x)\, dx.$$

We estimate the last two integrals in (8) as $r \to \infty$ and $\varepsilon \to 0$ respectively. Since

$$\left| \int_{S_r(0)} z^\alpha f(z)\, dz \right| \le \frac{A}{r^{2-\alpha}}\, 2r\pi$$

$$= \frac{2A\pi}{r^{1-\alpha}},$$

therefore, $$\lim_{r\to\infty}\int_{S_r(0)} z^{\alpha} f(z)\,dz = 0.$$

Also, since

$$\left|\int_{S_\varepsilon(0)} z^{\alpha} f(z)\,dz\right| \le \frac{A}{\varepsilon^{1-\alpha}}\,2\pi\varepsilon = 2A\pi\varepsilon^{\alpha},$$

hence, $$\lim_{\varepsilon\to 0}\int_{S_\varepsilon(0)} z^{\alpha} f(z)\,dz = 0.$$

Thus, $$(1-e^{2\pi i\alpha})\,I = 2\pi i\sum_{j=1}^{n}\operatorname{Res}\,(z^{\alpha} f(z), z_j).$$

Example: $$I = \int_0^{\infty}\frac{x^{\alpha}}{x\,(1+x)}\,dx \quad (0<\alpha<1).$$

We have

$$\begin{aligned}(1-e^{2\pi i\alpha})\,I &= 2\pi i\,\operatorname{Res}\left(\frac{z^{a}}{z(1+z)'}-1\right)\\ &= -\,2\pi i\,(-\,1)^{\alpha}\\ I &= \frac{2\pi i e^{\pi i\alpha}}{e^{2\pi i\alpha}-1} = \frac{2\pi i}{e^{\pi i\alpha}-e^{-\pi i\alpha}}\\ &= \frac{\pi}{\sin\pi\alpha}\quad (e^{\pi i} = -\,1).\end{aligned}$$

(*v*) Consider the integrals of the type

$$I = \int_0^{\infty} f(x)\log x\,dx$$

where $f(z) = g(z)/h(z)$ is a rational function, analytic on C except for finite number of poles, none of which lies on the positive real axis and the origin.

We assume that the degree of $h(z)$ is greater than the degree of $g(z)$ by at least two units.

Consider $\int_{\gamma} f(z)\log z\,dz$, where γ is the same closed curve as in Fig. 8.I. If $z_1, z_2, \ldots, z_r$ are singular points of $f(z)\log^2 z$, we have (as $\varepsilon\to 0$ and $r\to\infty$)

$$\begin{aligned}&\int_0^{\infty} f(x)\log^2 x\,dx - \int_0^{\infty} f(x)\,(\log x + 2\pi i)^2\,dx\\ &\qquad = 2\pi i\sum_{j=1}^{n}\operatorname{Res}\,(f(z)\log^2 z, z_j)\end{aligned}$$

i.e. $$\begin{aligned}&4\pi i\int_0^{\infty} f(x)\log x\,dx - 4\pi^2\int_0^{\infty} f(x)\,dx\\ &\qquad = -\,2\pi i\sum_{j=1}^{n}\operatorname{Res}\,(f(z)\log^2 z, z_j).\end{aligned}$$

If f is real-valued function on the real axis, we obtain

$$\int_0^{\infty} f(x)\log x\,dx = -\frac{1}{2}\operatorname{Re}\left\{\sum_{j=1}^{n}\operatorname{Res}\,(f(z)\log^2 z, z_j)\right\}$$

and $$\int_0^{\infty} f(x)\,dx = \frac{1}{2\pi}\,\text{Im}\left\{\sum_{j=1}^{n} \text{Res}\,(f(z)\log^2 z, z_j\right\}.$$

Example:

$$I = \int_0^{\infty} \frac{\log x}{1+x^2}\,dx.$$

The singularities of $\frac{1}{1+z^2}$ are i and $-i$, we have

$$\text{Res}\left(\frac{\log^2 z}{1+z^2}, i\right) = \frac{\pi^2}{8}\,i$$

and $$\text{Res}\left(\frac{\log^2 z}{1+z^2}, -i\right) = \frac{-9\pi^2}{8}\,i.$$

Thus $$I = \int_0^{\infty} \frac{\log x}{1+x^2}\,dx = 0$$

and $$\int_0^{\infty} \frac{dx}{1+x^2} = \pi/2.$$

8.3 THE LOGARITHMIC RESIDUE

Definition: Let Ω be an open set. Let f be meromorphic in Ω and $z_0 \in \Omega$. The **logarithmic residue** of f at z_0 is the residue of

$$\frac{f'}{f} = \frac{d}{dz}\{\log f(z)\}$$

evaluated at z_0.

Suppose that f is analytic in Ω and has a zero of order m at z_0. So

$$f(z) = (z-z_0)^m\, g(z) \text{ where } g(z_0) \neq 0.$$

Hence

$$(9) \quad \frac{f'}{f} = \frac{m}{z-z_0} + \frac{g'}{g}$$

and $\frac{g'}{g}$ is analytic in the neighbourhood of z_0. Now suppose that f has a pole of order m at z_0 ; *i.e.*

$f(z) = (z-z_0)^{-m}\, g(z)$ where g is analytic and $g(z_0) \neq 0$. It follows that

$$(10) \quad \frac{f'}{f} = \frac{-m}{z-z_0} + \frac{g'}{g}$$

and again $\frac{g'}{g}$ is analytic in the neighbourhood of z_0.

Now from (9) $\operatorname{Res}\left(\frac{f'}{f}, z_0\right) = \operatorname{Res}\left(\frac{m}{z - z_0}, z_0\right) + \operatorname{Res}\left(\frac{g'}{g}, z_0\right)$.

But $\operatorname{Res}\left(\frac{g'}{g}, z_0\right) = 0$ and therefore

$$\operatorname{Res}\left(\frac{f'}{f}, z_0\right) = m.$$

Similarly, from (10) we have

$$\operatorname{Res}\left(\frac{f'}{f}, z_0\right) = -m.$$

We summarize the above discussion in the following theorem.

Theorem 3. *Let f be meromorphic on Ω with poles $s_1, s_2, \ldots, s_m$ and zeroes $z_1, z_2, \ldots, z_n$. If γ is a closed chain in Ω homologous to zero in Ω and not passing through $s_1, s_2, \ldots, s_m; z_1, \ldots, z_n$,* then

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'}{f} = \sum_{k=1}^{n} n(\gamma, z_k) - \sum_{j=1}^{n} n(\gamma, s_j).$$

In applications, γ is taken as a circle or a rectangle, and the points $s_1, s_2, \ldots, s_m, z_1, z_2, \ldots, z_n$ lie inside γ.

Earlier we have defined "interior". In the following definition, we redefine "interior" which will be used in the subsequent discussion.

Definition: Let γ be a closed path. We say that γ has an interior if $n(\gamma, \alpha) = 0$ or 1 for every complex number α which does not lie on γ. The set of points α such that $n(\gamma, \alpha) = 1$ is called the interior of γ.

Theorem 4. (Rouche's Theorem). *Let γ be a closed path homologous to 0 in Ω. Assume that γ has an interior. Let f and g be analytic in Ω, and $|f(z) - g(z)| < |f(z)|$ for z on γ. Then f and g have the same number of zeroes in the interior of γ.*

Proof : It follows from the assumption that f and g have no zero on γ. We have

$$\left|\frac{g(z)}{f(z)} - 1\right| < 1 \text{ for } z \text{ on } \gamma.$$

This gives that the values of $\frac{g}{f}$ are contained in the open disk centred at 1 and radius

1. Let $h = \frac{g}{f}$. Then $h \circ \gamma$ is a closed path contained in that disk. Thus, since 0 lies outside the disk, we have

$$n(h \circ \gamma, 0) = 0.$$

If γ is defined on $[a, b]$ then

$$n\,(h \circ \gamma, 0) = \int_{h \circ \gamma} \frac{1}{z}\,dz$$

$$= \int_a^b \frac{h'(\gamma(t))}{h(\gamma(t))}\,\gamma'(t)\,dt = 0$$

i.e. $$\int_\gamma \frac{h'}{h} = 0$$

i.e. $$\int_\gamma \left(\frac{g'}{g} - \frac{f'}{f}\right) = 0.$$

It now follows from Theorem 3 that f and g have the same number of zeros. This completes the proof of Rouche's Theorem.

Rouche's theorem can be applied to give the proof of the Fundamental Theorem of Algebra. If $p\,(z) = z^n + a_1 z^{n-1} + \ldots + a_n$, then

$$\frac{p(z)}{z^n} = 1 + \frac{a_1}{z} + \ldots + \frac{a_n}{z^n}$$

and $$\lim_{z \to \infty} \left\{\frac{p(z)}{z^n}\right\} = 1.$$

So for sufficiently large number r we have

$$\left|\frac{p(z)}{z^n} - 1\right| < 1 \;(|z| = r)$$

i.e. $$|p(z) - z^n| < |z|^n \;(|z| = r).$$

It follows from Rouches' theorem that $p\,(z)$ must have n zeros inside $|z| = r$.

Example: Show that the equation

$e^z = az^n$ has n roots inside the unit circle, if $a > e$.

Write $f\,(z) = az^n$ and $g\,(z) = -\,e^z$. Then

$$f(z) + g(z) = az^n - e^x.$$

Suppose that γ is a positively oriented closed curve in a region Ω and we take γ to be a circle $|z| = 1$.

Then on γ,

$$|f(z)| = a$$

and $$|g(z)| = \left|1 + z + \frac{z^2}{2!} + \ldots\right|$$

$$\leq 1 + |z| + \frac{|z|^2}{2!} + \ldots$$

$$\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

$$= e.$$

Since $a > e$, it follows that $|g(z)| < |f(z)|$.

Applying Rouche's theorem we find that $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside $|z| = 1$. Since $f(z) = az^n$ has n zeros, all at the origin, it follows from the Fundamental Theorem of Algebra that $e^z = ez^n$ has n roots inside $|z| = 1$.

EXERCISES

1. Prove that

(*i*) $\operatorname{Res}\left(\frac{z}{\sin z}, 0\right) = 0$ (*ii*) $\operatorname{Res}(\tan z, \pi/2) = -1$

(*iii*) $\operatorname{Res}\left(\frac{\sin z}{z^2}, 0\right) = 1$ (*iv*) $\operatorname{Res}\left(\frac{1-\cos z}{z^2}, 0\right) = \frac{1}{2}$

2. Compute the following integrals where C_1 is the unit circle (counterclockwise)

(*i*) $\int_{C_1} \frac{dz}{z^2 + 6iz}$ (*ii*) $\int_{C_1} \frac{e^z}{\sin z}\, dz$

(*iii*) $\int_{C_1} \frac{e^z}{\cos \pi z}\, dz$ (*iv*) $\int_{C_1} \frac{(z^2+2)^3}{2z^2 - z}\, dz$

(*v*) $\int_{C_1} \frac{(z+4)^3}{z^4 + 5z^3 + 6z^2}\, dz$

3. Evaluate the following integrals

(*i*) $\int_{-\infty}^{\infty} \frac{dx}{x^6 + 1}$ (*ii*) $\int_{-\infty}^{\infty} \frac{x^2}{x^4 + 1}\, dx$

(*iii*) $\int_0^{\infty} \frac{x^2}{x^6 + 1}\, dx$ (*iv*) $\int_0^{\infty} \frac{x^2\, dx}{(x^2 + 1)^2}$

4. Compute the following integrals

(*i*) $\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + 1}\, dx$ (*ii*) $\int_{-\infty}^{\infty} \frac{\sin x}{x^3 + 1}\, dx$

(*iii*) $\int_0^{\infty} \frac{\cos x}{(x^2 + 1)^2}\, dx.$

5. Prove that

(*i*) $\int_0^{2\pi} \frac{d\theta}{1 + \sin^2 \theta} = \pi\sqrt{2}$ (*ii*) $\int_0^{2\pi} \frac{\sin^2 \theta}{1 + \frac{1}{2}\cos \theta}\, d\theta = 4\pi(2 - \sqrt{3})$

(*iii*) $\displaystyle\int_0^{\pi} \frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2} = \frac{8\pi}{3\sqrt{3}}$ (*iv*) $\displaystyle\int_0^{\pi/2} \frac{d\theta}{1+\sin^2\theta} = \frac{\pi}{2\sqrt{2}}$

(*v*) $\displaystyle\int_0^{\pi} \frac{d\theta}{3+2\cos\theta} = \frac{\pi}{\sqrt{5}}$ (*vi*) $\displaystyle\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta} = \frac{\pi}{\sqrt{1+a^2}}$

6. Show that

$$\int_0^{\pi/2} \frac{d\theta}{(a+\sin^2\theta)^2} = \frac{\pi\,(2a+1)}{4\,(a^2+a)^{3/2}}, (a>0)$$

7. Prove that

$$\int_0^{2\pi} \frac{d\theta}{(2-\sin\theta)} = 2\pi/\sqrt{3}$$

8. Prove that

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}, (0<b<a)$$

9. Prove that

(*i*) $\displaystyle\int_0^{\infty} \frac{\log x}{(1+x^2)^2}\,dx = -\pi/4$ (*ii*) $\displaystyle\int_0^{\infty} \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi\log 2}{2}$

(*iii*) $\displaystyle\int_0^{\infty} \frac{(\log x)^2}{1+x^2}\,dx = \pi^2/8$ (*iv*) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \frac{\pi}{\sin a\pi}, 0<a<1$

10. Compute

$$\int_{\gamma} \frac{e^{-z^2}}{z^2}\,dz$$

where γ is : (*a*) The square with vertices $1+i, -1+i, -1-i, 1-i$

(*b*) The ellipse defined by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

11. Compute

$$\int_0^{\infty} \frac{z^{2m}}{1+x^{2n}}\,dx \text{ for } n>m\geq 0$$

12. Prove that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^n} = \frac{\pi}{2^{n-2}}\cdot\frac{(2n-2)\,!}{[(n-1)\,!]^2}$$

13. Prove that

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = \pi e^{-a} \text{ if } a>0.$$

14. Prove that for any real number $a>0$.

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\, dx = \pi e^{-a} / a$$

15. Prove that

$$(i) \quad \int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)}\, dx = \frac{\pi\,(1+a)}{2a^3\, e^a},\ (a > 0)$$

$$(ii) \quad \int_{0}^{\infty} \frac{\cos x}{(x^2 + a^2)\,(x^2 + b^2)}\, dx = \frac{\pi}{a^2 - b^2}\left(\frac{1}{be^b} - \frac{1}{ae^a}\right),\ (0 < b < a)$$

16. Prove that

$$\int_{0}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \pi/2$$

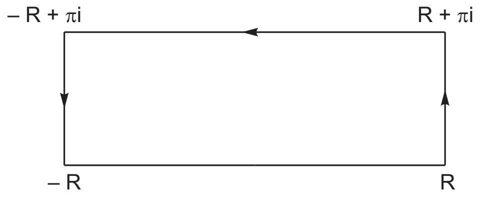
17. Prove that

$$\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2}\, dx = \frac{\pi \sin a}{a},\ (a > 0)$$

18. Prove that

$$\int_{-\infty}^{\infty} \frac{\cos x}{e^x + e^{-x}}\, dx = \frac{\pi}{e^{\pi/2} - e^{-\pi/2}}$$

Use the indicated contour:



19. Prove that

$$(i) \quad \int_{0}^{\infty} \frac{x^e}{1+x}\, \frac{dx}{x} = \frac{\pi}{\sin \pi a},\ (0 < a < 1)$$

$$(ii) \quad \int_{0}^{\infty} \frac{x^a}{1+x^3}\, \frac{dx}{x} = \frac{\pi}{3 \sin\left(\frac{\pi a}{3}\right)},\ (0 < a < 3).$$

20. Show that

$$\int_{0}^{2\pi} \frac{d\theta}{1 + a^2 - 2a\cos\theta} = \frac{2\pi}{1 - a^2},\ (0 < a < 1).$$

9

CONFORMAL MAPPING

The main general theorem concerning **conformal mapping** is the **Riemann Mapping** Theorem:

"*If G is a simply connected region which is not the whole plane, then there exists an isomorphism of G with the open disk $D_1(0)$*".

The proof of this theorem is postponed to a later chapter (see Appendix 1). In this chapter, we shall give specific examples, where the mapping can be exhibited concretely. We begin with a simple example.

Consider the function defined by

$$w = f(z) = z^2.$$

Put $z = re^{i\theta}$ and $w = Re^{i\phi}$

Then $$R = r^2 \text{ and } \phi = 2\theta.$$

We find that circles r = const. are mapped onto circles R = const. and **rays** $\theta = \theta_0$ = const. onto rays $\varphi = 2\theta_0$ = const. In particular, the positive real axis ($\theta = 0$) in the z-plane is mapped onto the positive real axis in the w-plane, and the positive imaginary axis ($\theta = \pi/2$) in the z-plane is mapped onto the negative real axis in the w-plane. The angles at the origin are doubled under the mapping. The first quadrant $0 \leq \theta \leq \pi/2$ is mapped onto the entire upper half of the w-plane.

Let $z = x + iy$ and $w = u + iv$. Then $u = x^2 - y^2$, $v = 2xy$. Hence, the hyperbolas $x^2 - y^2 = c$ and $2xy = d$ are mapped by f into the straight lines $u = c$, $v = d$. One interesting fact is that for c and d not zero, these hyperbolas intersect at right angles, just as their images do.

Now examine what happens to the lines $x = c$ and $y = d$. Consider $x = c$ where y is arbitrary. f maps $x = c$ into $u = c^2 - y^2$ and $v = 2cy$. Eliminating y we get that $x = c$ is mapped onto the parabola $v^2 = -4c^2(u - c^2)$. Similarly, f maps the line $y = d$ onto the parabola $v^2 = 4d^2(u + d^2)$.

The above discussion sheds some light on the nature of $f(z) = z^2$ and likewise, it is useful to study the mapping properties of other analytic functions.

Suppose $\gamma : [a, b] \rightarrow G$ is a differentiable path and that for some t_0 in (a, b), $\gamma'(t_0) \neq 0$. Then γ has a tangent at the point $\gamma(t_0)$. The slope of the line is tan (arg $\gamma'(t_0)$). If γ_1 and γ_2 are two paths with $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ (say) and $\gamma_1'(t_1) \neq 0$, $\gamma_2'(t_2) \neq 0$, then define the angle between the paths γ_1 and γ_2 at z_0 to be arg $\gamma_2'(t_2)$ – arg $\gamma_1'(t_1)$.

Suppose γ is a path in G and $f : G \rightarrow C$ is analytic. Then $\sigma = f \circ \gamma$ is also a path and $\sigma'(t) = f'(\gamma(t))\, \gamma'(t)$.

Let $z_0 = \gamma(t_0)$ and suppose that $\gamma'(t_0) \neq 0$ and $f'(z_0) \neq 0$, then $\sigma'(t_0) \neq 0$ and

arg $\sigma'(t_0)$ = arg $f'(z_0)$ + arg $\gamma'(t_0)$, *i.e.*

(1) $\quad$ arg $\sigma'(t_0)$ – arg $\gamma'(t_0)$ = arg $f'(z_0)$.

Now let γ_1 and γ_2 be paths with $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ (say) and $\gamma_1'(t_2) \neq 0$, $\gamma_2'(t_2) \neq 0$. Let $\sigma_1 = f \circ \gamma_1$ and $\sigma_2 = f \circ \gamma_2$. Suppose also that the paths γ_1 and γ_2 are not tangent to each other at z_0. In other words, suppose that $\gamma_1'(t_1) \neq \gamma_2'(t_2)$. Equation (1) yields

(2) $\quad$ arg $\gamma_2'(t_2)$ – arg $\gamma_1'(t_1)$ = arg $\sigma_2'(t_2)$ – arg $\sigma_1'(t_1)$.

Thus given any two paths through z_0, f maps these paths onto two paths through $w_0 = f(z_0)$, and when $f'(z_0) \neq 0$, the angles between the curves are preserved both in magnitude and direction. We summarize the above discussion in the following theorem.

Theorem 1. Let $f : G \rightarrow C$ *be analytic. Then f preserves angles at each point* z_0 *of G where* $f'(z_0) \neq 0$.

9.1 DEFINITION

Let $f : G \rightarrow C$ be such that it preserves angles and

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} \text{ exists,}$$

then f is called a **conformal map.** If f is analytic and $f'(z) \neq 0$ for any z, then f is **conformal.** The converse of this statement is also true.

Example 1: The mapping $w = f(z) = z^2$ is conformal except at $z = 0$. At $z = 0$, the angles are doubled under the mapping, because each ray arg $z = c$ = const. maps into a ray arg $w = 2c$ = const. as in Fig. 9.I.

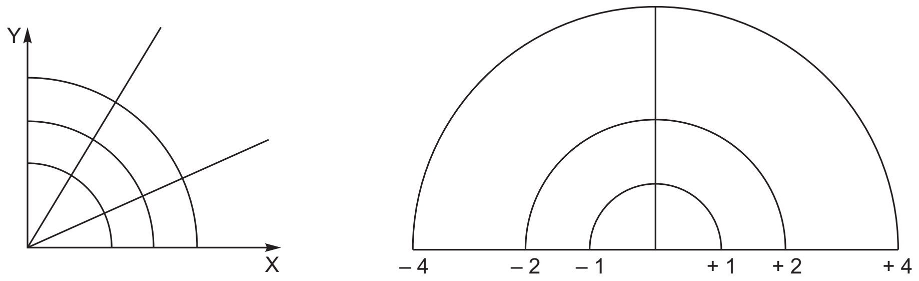


Fig. 9.I. Mapping $w = z^2$.

Example 2: The mapping $w = f(z) = e^z$ is conformal throughout C.

In order to discuss the other properties of this mapping put $z = c + iy$ where c is fixed, then $f(z) = re^{iy}$ for $r = e^c$. That is, f maps the line $x = c$ onto the circle with centre at the origin and radius e^c. Also, f maps the line $y = d$ onto the **infinite ray** $\{re^{id} : 0 < r < d\}$.

We now find the image of the vertical line segment

$$x = c, -\pi < y \leq \pi$$

under the mapping given by $w = f(z) = e^z$.

Note that every point on the given line segment has the form

$$z = c + iy, (-\pi < y \leq \pi)$$

hence, as y varies from $-\pi$ to $+\pi$, e^{iy} describes a complete circle, while $|w|$ remains fixed at e^c. In other words, as z varies on the given line segment, w describes a circle centred at $w = 0$ and having radius e^c (see Fig. 9.II).

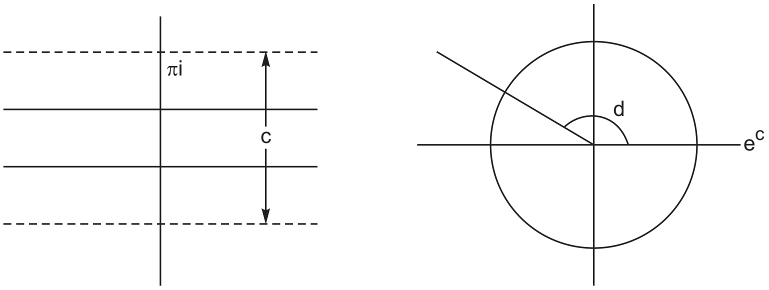


Fig. 9.II

Observe that if y were allowed to vary over a larger domain (but, always, on the same vertical line), then w would repeat its trace on the same circle, and if we took the entire vertical line $x = c$, then the circle $|w| = e^c$ would be repeated infinitely many times.

We summarize as follows:

If we take all the horizontal lines between $y = -\pi$ (not included) and $y = \pi$ (inclusive), their images will be all the rays with angles of inclination ranging from $-\pi$ to π. The totality of all such rays cover all the points in the w-plane except $w = 0$. On the other hand, if we take all the vertical line segments, as in the above example, contained between the lines $y = -\pi$ and $y = \pi$, then their images will be all the circles, with positive radius centred at $w = 0$. The totality of all such circles will cover all the points in the w-plane except $w = 0$.

From the preceding discussion we arrive at the following conclusion:

Under the mapping $w = f(z) = e^z$, the fundamental strip

$$S : -\pi < y \leq \pi, -\infty < x < +\infty,$$

is mapped onto the entire w-plane except its origin. Arguing similarly we can say that any horizontal strip

$$S_\alpha : \alpha < y < \alpha + 2\pi, -\infty < x < +\infty \text{ (α is any real number)}$$

is mapped onto the entire w-plane except its origin.

Note that the function log z is the inverse of the function e^z and this means that log z maps the z-plane (minus its origin) onto the fundamental strip

$$-\pi < v \leq \pi, -\infty < u < +\infty$$

of the w-plane. We shall arrive at the same conclusion as above by considering the logarithmic transformation directly.

Consider

$$w = \log |z| + i \arg z, z \neq 0$$

Then

$$u = \log |z|, v = \arg z.$$

Now, as z varies over all non-zero values, $|z|$ varies between 0 and $+\infty$, hence $\log |z|$ varies from $-\infty$ to $+\infty$, and, therefore, $-\infty < u < +\infty$. On the other hand, since, by definition, $-\pi < \arg z \leq \pi$, we have $-\pi < v \leq \pi$. The last two relations involving u and v represent the fundamental strip of the w-plane.

9.2 LINEAR FRACTIONAL TRANSFORMATION

We shall now consider an important class of conformal mappings, starting with the simplest types of transformations in this class.

The mapping

(3) $$w = h(z) = z + b$$

is called a **translation**. If $w = h(z) = az$ with $a \neq 0$, then the mapping is called a **dilation.** If $w = h(z) = e^{i\theta} . z$, then the mapping is called a **rotation.** If $w = h(z) = 1/z$, then it is called the **inversion.**

A mapping of the form

(4) $$w = h(z) = \frac{az + b}{cz + d}$$

is called a **linear fractional transformation.** If a, b, c and d also satisfy $ad - bc \neq 0$ then the mapping is called a **Möbius transformation.**

If h is a Möbius transformation then $h^{-1}(z) = \dfrac{dz - b}{-cz + a}$ satisfies

$$h(h^{-1} z) = h^{-1}(hz) = z.$$

If h and g are both linear fractional transformations then it follows that $h \circ g$ is also a linear fractional transformation.

Consider the Möbius transformation

$$h(z) = \frac{az+b}{cz+d}.$$

Then

$$h'(z) = \frac{ad-bc}{(cz+d)^2}.$$

This shows that the condition $ad - bc \neq 0$ implies that $h^1(z) \neq 0$. Hence, the mapping h is conformal.

It follows from (4) that to each z for which $cz + d \neq 0$ there corresponds precisely one complex number $h(z)$. Suppose that $c \neq 0$. Then to each $z = -d/c$ where $cz + d \neq 0$, there does not correspond a number $h(z)$. Thus we attach a point ∞ to the w-plane. We denote $C_\infty = C \cup \{\infty\}$. We consider h defined on C_∞ where $h(\infty) = a/c$ and $h(-d/c) = \infty$. Note that when $a = 0 = c$ or $d = 0 = c$ then $ad - bc = 0$, hence we cannot take $a = 0 = c$ or $d = 0 = c$. Since h has an inverse it maps C_∞ onto C_∞.

Theorem 2. *If h is a Möbius transformation then h is the composition of transformations of the types* (*i*) *translation,* (*ii*) *dilation and* (*iii*) *inversion.* (*Of course, some of these may be missing.*)

Proof : We have

$$h(z) = \frac{az+b}{cz+d}, \text{ where } ad - bc \neq 0.$$

Suppose that $c = 0$. Hence

$$h(z) = \left(\frac{a}{d}\right)z + \left(\frac{b}{d}\right).$$

If $$h_1(z) = \left(\frac{a}{d}\right)z,\ h_2(z) = z + \left(\frac{b}{d}\right),$$

then $$h_2 \circ h_1 = h.$$

Suppose now that $c \neq 0$ and put

$$h_1(z) = z + \frac{d}{c},\ h_2(z) = \frac{1}{z},$$

$$h_3(z) = \frac{bc-ad}{c^2}z,\ h_4(z) = z + \frac{a}{c}.$$

Then $$h = h_4 \circ h_3 \circ h_2 \circ h_1.$$

The **fixed points** of the mapping (4) are obtained from the equation

$$z = \frac{az+b}{cz+d}$$

i.e. $$cz^2 + (d-a)z - b = 0.$$

Hence, a Möbius transformation can have at most two fixed points unless $h(z) = z$ for all z.

Now let h be a Möbius transformation and let a, b, c be distinct points in C_∞ with $\xi = h(a)$, $\eta = h(b)$, $\zeta = h(c)$. Suppose that g is another transformation with this property. Then g^{-1} o h has fixed points a, b, c. Hence g^{-1} o $h = I =$ the **identity transformation.** Thus a Möbius transformation is uniquely determined by its action on three given points in C_∞.

Theorem 3. *A Möbius transformation takes circles onto circles.*

Proof: It can be checked easily that the theorem holds for the transformation of the type $w = az + b$. We show that the theorem also holds for the transformation of the type $w = \frac{1}{z}$.

The family of circles is represented by the equations

$$(5) \qquad \alpha z\bar{z} + \beta z + \overline{\beta z} + \gamma = 0$$

where α and γ are real. If $\alpha = 0$, then (5) reduces to a straight line. By using the transformation $w = \frac{1}{z}$ we find that (5) is mapped onto

$$(6) \qquad \alpha + \beta w + \overline{\beta w} + \gamma w\overline{w} = 0.$$

When $\gamma \neq 0$, (6) represents a circle and when $\gamma = 0$, (6) represents a straight line. Theorem 3 is then a consequence of Theorem 2.

Example: Show that Möbius transformation $w = \frac{z-1}{z+1}$ maps the half-plane Re $z > 0$ onto the unit disc $|\, w\,| < 1$.

We discuss this mapping in three gradual steps, each of which is accompanied by a drawing in Fig. 9.IIA. It is easy to see that the given map is obtained by successive application of the maps:

$$\zeta = z + 1,\ \eta = \frac{1}{\zeta},\ w = 2\eta + 1.$$

Step I : Under the map $\zeta = z + 1$, Re $z > 0$ is mapped onto the half-plane Re $\zeta > 1$; see Fig. 9IIA (*a*).

Step II : Under the map $\eta = \frac{1}{\zeta}$, the half-plane Re $\zeta > 1$ is mapped onto the interior of the circle $\left|\eta - \frac{1}{2}\right| = \frac{1}{2}$, but with the upper and lower halves of the half-plane and of the circle interchanged, see Fig. 9.IIIA (*b*).

Step III : Under the map $w = -2\eta + 1$, the interior of the circle found in step II will be rotated through $-\pi$ radians onto the interior of the circle $\left|\eta - \frac{1}{2}\right| = \frac{1}{2}$, but the rotation will interchange the positions of the upper and lower halves of the disc. Then the rotation will be followed by a stretching (by a factor of 2) onto the interior of the circle $| \eta + 1 | = 1$, and, lastly, a shift through the vector 1 will yield the disc $| w | < 1$.

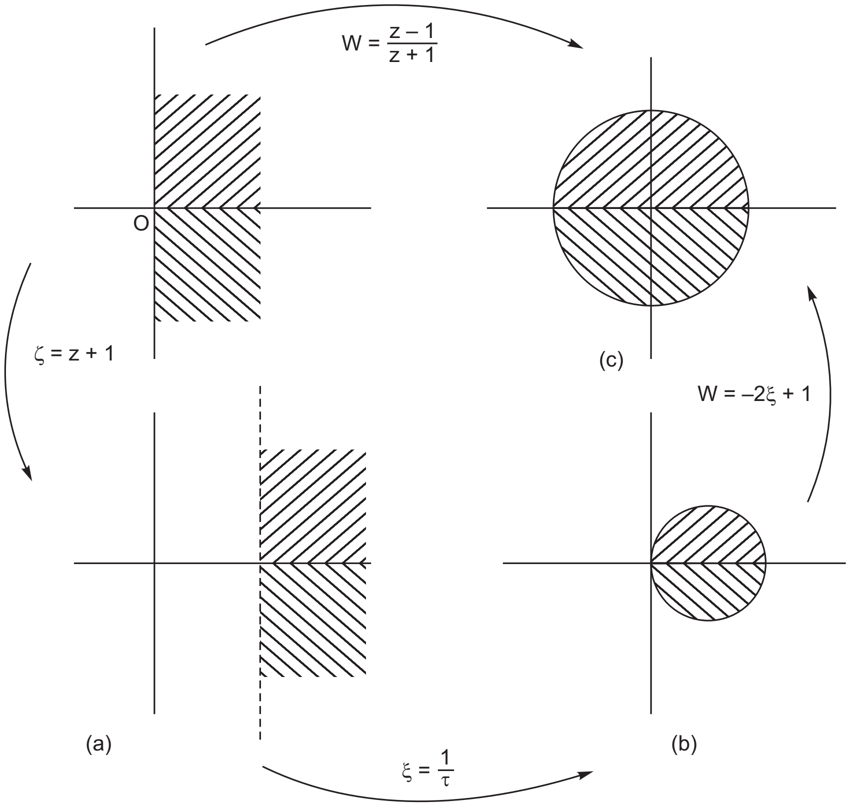


Fig. 9.III

9.3 DEFINITION

Let z_2, z_3, z_4 be points in C_∞. We define

$$M : C_\infty \to C_\infty \text{ by}$$

$$M(z) = \left(\frac{z - z_3}{z - z_4}\right) \Big/ \left(\frac{z_2 - z_3}{z_2 - z_4}\right).$$

If z_2, z_3, z_4 are points in C

$$M(z) = \frac{z - z_3}{z - z_4}, \text{ if } z_2 = \infty;$$

$$M(z) = \frac{z_2 - z_4}{z - z_4}, \text{ if } z_3 = \infty;$$

$$M(z) = \frac{z - z_3}{z_2 - z_3}, \text{ if } z_4 = \infty.$$

Observe that M is the only transformation where $M(z_2) = 1$, $M(z_3) = 0$, $M(z_4) = \infty$.

Definition

The **cross-ratio** of four points is defined by

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}.$$

Theorem 4. *A Möbius transformation leaves a cross-ratio invariant.*

This can be verified first for $w = az + b$, and then for $w = \frac{1}{z}$.

Theorem 4 permits us to write down the Möbius transformation which maps the points z_1, z_2, z_3 into the points w_1, w_2, w_3 :

$$\frac{w - w_1}{w - w_2} \cdot \frac{w_2 - w_3}{w_1 - w_3} = \frac{z - z_1}{z - z_2} \cdot \frac{z_2 - z_3}{z_1 - z_3}.$$

Example 3: Find the Möbius transformation which maps the points $z = -1$, 0, 1 into the points $w = 0, i, 3i$, respectively.

Setting up the appropriate cross-ratios, we have

$$\frac{(w - 0)}{(w - i)} \cdot \frac{(i - 3i)}{(0 - 3i)} = \frac{(z + 1)}{(z - 0)} \cdot \frac{(0 - 1)}{(-1 - 1)}$$

After a little calculation, we get

$$w = -3i\frac{z + 1}{z - 3}.$$

9.4 SYMMETRY

We recall that the union $C \cup \{\infty\}$ is the extended complex plane. Denote the extended complex plane by C_∞.

Let Γ be a circle through points z_2, z_3, z_4. We say that the points z, z^* in C_∞ are **symmetric** with respect to Γ if

$$(7) \qquad (z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}.$$

We explain below what it means for z and z^* to be symmetric.

Let Γ be a straight line. Choose $z_4 = \infty$. Then (7) reduces to

$$\frac{z^* - z_3}{z_2 - z_3} = \frac{\bar{z} - \bar{z}_3}{\bar{z}_2 - \bar{z}_3}.$$

That is,

$$| z^* - z_3 | = | z - z_3 |.$$

Hence, z and z^* are equidistant from each point on the straight line Γ.

Also, $$\operatorname{Im} \frac{z^* - z_3}{z_2 - z_3} = \operatorname{Im} \frac{\bar{z} - \bar{z}_3}{\bar{z}_2 - \bar{z}_3} = -\operatorname{Im} \frac{z - z_3}{z_2 - z_3}.$$

Hence, z and z^* lie in different half planes determined by Γ.

Thus, the line segment $[z, z^*]$ is perpendicular to Γ as in Fig. 9.IV.

Suppose that Γ is a circle, *i.e.*

$$\Gamma = \{z : |z - \alpha| = r\} \ (0 < r < \infty).$$

Let Γ passes through the points z_2, z_3, z_4. It follows from equation (7) that

$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}.$$

Since the Möbius transformations leave the cross-ratio invariant, we have

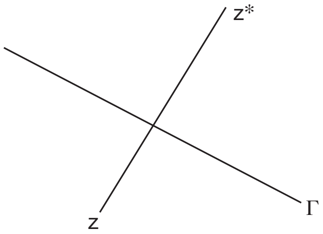


Fig. 9.IV

$$\begin{aligned}\overline{(z, z_2, z_3, z_4)} &= \overline{(z-\alpha, z_2-\alpha, z_3-\alpha, z_4-\alpha)}\\ &= \left(\bar{z}-\bar{\alpha}, \frac{r^2}{z_2-\alpha}, \frac{r^2}{z_3-\alpha}, \frac{r^2}{z_4-\alpha}\right)\\ &= \left(\frac{r^2}{\bar{z}-\bar{\alpha}}, z_2-\alpha, z_3-\alpha, z_4-\alpha\right)\\ &= \left(\frac{r^2}{\bar{z}-\bar{\alpha}}+\alpha, z_2, z_3, z_4\right)\end{aligned}$$

Hence, $$z^* = \frac{r^2}{\bar{z}-\bar{\alpha}} + \alpha$$

i.e. $$(z^* - \alpha)(\bar{z} - \bar{\alpha}) = r^2.$$

From this we have

$$\frac{z^*-\alpha}{z-\alpha} = \frac{r^2}{|z-\alpha|^2} > 0.$$

Thus z^* lies on the ray $\{\alpha + \lambda(z - \alpha) : 0 < \lambda < \alpha\}$. We illustrate this by a diagram (Fig. 9.V).

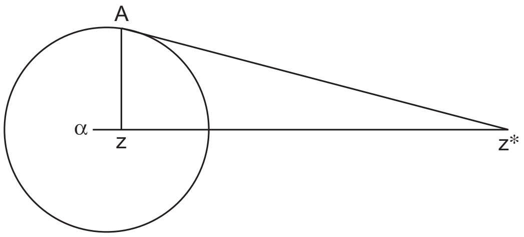


Fig. 9.V

Let L be the ray from α through z. Let P be the perpendicular line to L at z and let P intersects Γ at A. Draw a tangent at A. Then the point of intersection of this tangent with L is the point z^*.

Theorem 5 (The Principle of Symmetry). *The Möbius transformations preserve the symmetry ; that is, if a Möbius transformation M maps a circle* C_1 *onto the circle* C_2, *the z and z* symmetric with respect to* C_1 *are mapped onto w and w* symmetric with respect to* C_2.

Proof : It follows from Theorem 3 that the circles Γ_1 passing though z and z^* are mapped onto the circles Γ_2 passing through w and w^*. Since the Möbius transformations are conformal, the result follows.

Other Examples

Example 4: Find the Möbius transformation which maps the upper half of the z-plane into the interior of the unit circle in the w-plane (Fig. 9.VI).

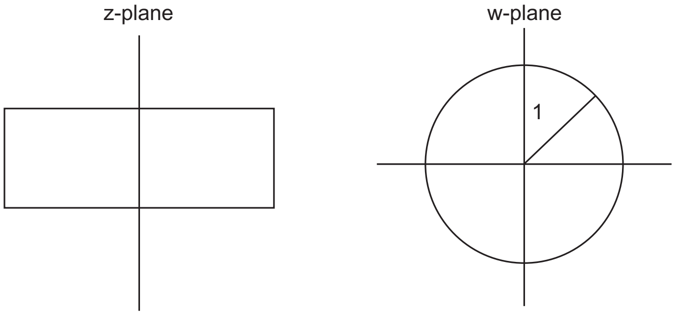


Fig. 9.VI

Let $$w = \frac{az+b}{cz+d}$$

be the required transformation. Since the unit circle in the w-plane is the image of the real axis in the z-plane, we have

$$|w| = \frac{|a|}{|c|} \cdot \frac{\left| z + \frac{b}{a} \right|}{\left| z + \frac{d}{c} \right|} = 1.$$

When $|z| \to \infty$, we get $\dfrac{|a|}{|c|} = 1.$

That is, $\dfrac{a}{c} = e^{i\theta}$ and we have for all real values of z

$$\left| z - \left(-\frac{b}{a} \right) \right| = \left| z - \left(-\frac{d}{c} \right) \right|.$$

This is possible iff the real axis in the z-plane is the perpendicular bisector of the line joining the points $-\dfrac{b}{a}$ and $-\dfrac{d}{c}$.

Thus $-\dfrac{b}{a}$ and $-\dfrac{d}{c}$ are conjugate complex numbers and we denote them by λ and $\bar{\lambda}$.

Hence we have

$$w = \frac{a}{c} \cdot \frac{z-\lambda}{z-\bar{\lambda}}$$

$$= e^{i\theta}\,\frac{z-\lambda}{z-\bar{\lambda}}. \tag{8}$$

We can easily check that the upper half of the z-plane is not mapped onto the outside of $|\,w\,| = 1$. The point $z = \lambda$ is mapped into the point $w = 0$, which is inside the unit circle $|\,w\,| = 1$. This completes the solution.

As a special case, let $e^{i\theta} = -1$, and let $\lambda = i$. Then

$$w = -\frac{z-i}{z+i}. \tag{9}$$

Now $$\operatorname{Im}(w) = \frac{w-\bar{w}}{2i}$$

$$= -\frac{1}{2i}\left(\frac{z-i}{z+i} - \frac{\bar{z}+i}{\bar{z}-i}\right).$$

That is, we have

$$\operatorname{Im}(w) = \frac{z+\bar{z}}{(z+i)(\bar{z}-i)}. \tag{10}$$

The denominator of the right side of (10) is a positive number. Thus $I(w)$ is positive iff $z+\bar{z}$ is positive. But $z+\bar{z} = 2\operatorname{Re}(z)$. This shows that the transformation (9) maps the upper half of the z-plane onto the unit circle in the w-plane in such a way that the first quadrant of the z-plane ($\operatorname{Re}(z) > 0$) is mapped onto the upper half of the circle ($\operatorname{Im}(w) > 0$) and the second quadrant of the z-plane is mapped onto the lower half of the circle.

The inverse transformation

$$z = -i\,\frac{w-1}{w+1} \tag{11}$$

maps the interior of the unit circle in the w-plane onto the upper half of the z-plane in such a way that the upper half of the circle is mapped onto the first quadrant of the z-plane.

Example 5: Find a transformation which maps a sector of angle $\pi/4$ onto the interior of the unit circle.

In fact, we cannot get the result only by taking the Möbius transformation. The result follows if we take the composition of transformations, namely

$$t = z^4 \quad \text{and} \quad w = \frac{t-i}{t+i}.$$

That is, $$w = \frac{z^4 - i}{z^4 + i}$$

gives the desired result as illustrated in the Fig. 9.VII.

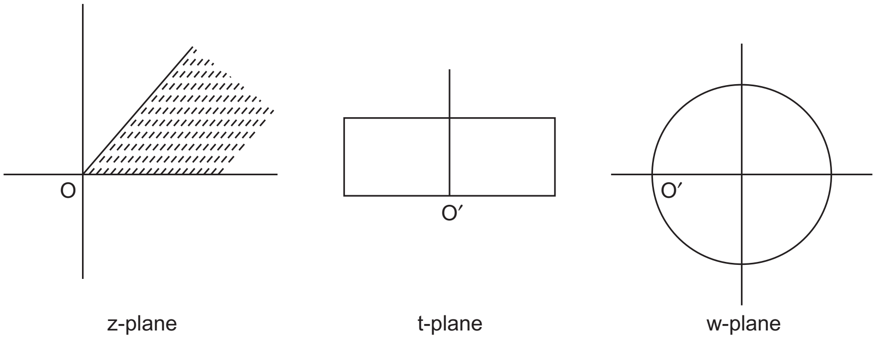


Fig. 9.VII

9.5 THE SCHWARZ-CHRISTOFFEL TRANSFORMATION

Consider a polygon [Fig. 9.VIII] in the w-plane having vertices at $w_1, w_2, ..., w_n$ and $\alpha_1, \alpha_2, ..., \alpha_n$ the corresponding interior angles. Let the points $w_1, w_2, ..., w_n$ map respectively into the points $x_1, x_2, ..., x_n$ on the real axis of z-plane (Fig. 9.VIII).

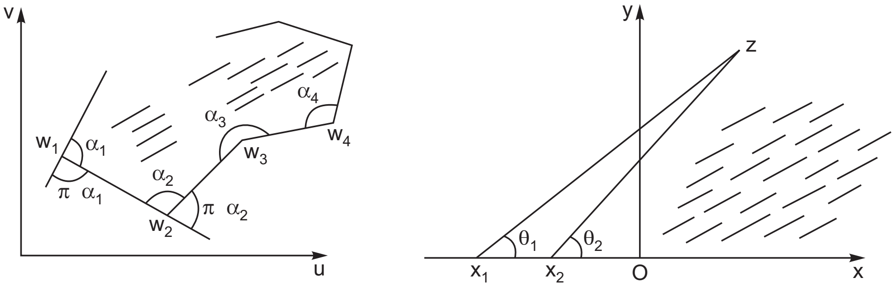


Fig. 9.VIII

The transformation

$$(12) \qquad w = K \int (z - x_1)^{\alpha_1/\pi - 1} (z - x_2)^{\alpha_2/\pi - 1} \ldots (z - x_n)^{\alpha_n/\pi - 1}$$

(where K is a complex constant) can be shown to map the interior of the polygon in the w-plane onto the upper half of the z-plane. The transformation can also be written as

$$(13) \qquad \frac{dw}{dz} = K (z - x_1)^{\alpha_1/\pi - 1} (z - x_2)^{\alpha_2/\pi - 1} \ldots (z - x_n)^{\alpha_n/\pi - 1}$$

Any three of the points $x_1, x_2, ..., x_n$ can be chosen at will. If x_n is chosen at infinity the last factor in (12) and (13) is not present. Note that infinite open polygons can be considered as limiting cases of closed polygons. The points $w_1, w_2, ...$ are mapped onto the points $x_1, x_2, ...$ on the real axis. It follows from (13) that

$$(14)\qquad \arg(dw) = \arg(dz) + \arg(K) + \left(\frac{\alpha_1}{\pi} - 1\right)\arg(z - x_1)$$

$$+ \left(\frac{\alpha_2}{\pi} - 1\right)\arg(z - x_2) + \ldots \left(\frac{\alpha_n}{\pi} - 1\right)\arg(z - x_n).$$

Suppose that as z in the z-plane moves along the real axis from the left toward x_1, the point w in the w-plane moves along a side of the polygon toward w_1. When z crosses from the left of x_1 to the right of x_1, $\theta_1 = \arg(z - x_1)$ changes from π to 0 while all other terms in (14) remain unchanged. Hence arg (dw) decreases by $(\alpha_1/\pi - 1)\arg(z - x_1) = \alpha_1 - \pi$. In other words, the direction through the point w_1 turns through the angle $\pi - \alpha_1$. Similarly, as z moves through x_2, $\theta_1 = \arg(z - x_1)$ and $\theta_2 = \arg(z - x_2)$ change from π to 0 while other terms in (14) remain constant. Hence w_2 turns through the angle $\pi - \alpha_2$ in the w-plane. By continuing the process, we see that as z moves along the real axis in the z-plane the point w moves along the sides of the polygon in the w-plane and vice-versa.

It can be shown that the interior of the polygon is mapped onto the upper half plane by (13). For this purpose, it is sufficient to prove that the transformation maps the interior onto the unit circle. Suppose that the transformation $w = f(z)$ maps the polygon P onto the unit circle $|z| = 1$ in the z-plane and that $f(z)$ is analytic inside and on $|z| = 1$. Now we have to show that to each point a inside the polygon P there corresponds one and only one point z_0 such that $f(z_0) = a$.

By Cauchy's integral formula

$$\frac{1}{2\pi i}\int_P \frac{dw}{w - a} = 1.$$

Since $w - a = f(z) - a$, we have

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{f'(z)}{f(z) - a}\,dz = 1.$$

But $f(z) - a$ is analytic inside $|z| = 1$. Hence, it follows from Theorem 3, § 7.3 that there is only one zero of $f(z) - a$ inside $|z| = 1$. In other words, $f(z_0) = a$. Observe that in order to get the required result we have used the known fact that the unit circle can be mapped onto the upper half plane.

9.6 THE TRANSFORMATIONS *w* = sin z AND *w* = cos *z*

Consider the interval of the real axis described by

$$-\pi/2 \le x \le \pi/2,\ y = 0$$

Consider the mapping $w = \sin z$. From the decomposition we have

$$w = \sin x \cosh y + i \sinh y \cos x$$

Then

$$(15)\qquad u = \sin x \cosh y \text{ and } v = \cos x \sinh y$$

If $y = 0$, then $\cosh y = 1$ and $\sinh y = 0$. It follows that for any point on the given interval, the last equations become

$$u = \sin x \text{ and } v = 0$$

Now, as x varies between $-\pi/2$ and $\pi/2$, $\sin x$ and therefore u varies between -1 and $+1$. Hence, under the mapping $w = \sin z$, the given interval maps onto the interval

$$-1 \leq u \leq 1,\ v = 0, \text{ in the } w\text{-plane.}$$

We now show that the imaginary axis $x = 0$ is mapped onto the imaginary axis $u = 0$, under the mapping $w = \sin z$.

We have from (15)

$$u = \sin x \cosh y,\ v = \cos x \sinh y$$

When $x = 0$, $u = 0$ and $v = \sinh y$. Then, as y varies from $-\infty$ to $+\infty$ on the imaginary axis, $\sinh y$ and hence v varies from $-\infty$ to $+\infty$, while u remains fixed at 0. This shows that the axis $x = 0$ is mapped onto the axis $u = 0$.

We now consider the transformation $w = \cos z$. Using the decomposition

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

We can consider several special cases as considered for $\sin z$.

But $$\cos z = \sin (z + \pi/2)$$

Hence, we can translate what we know about $\sin z$ into mapping properties of $\cos z$. In other words, $w = \cos z$ can be expressed as the composite mappings:

$$\zeta = z + \pi/2,\ w = \sin \zeta$$

We illustrate this process in the following example.

Consider the interval

$$I : -\pi \leq x \leq 0,\ y = 0.$$

Under the mapping $\zeta = z + \pi/2$, I is mapped onto the interval

$$J : -\pi/2 \leq R(\zeta) \leq \pi/2,\ \text{Im}(\zeta) = 0$$

Then, using the result of the above example, we see that, under the mapping $w = \sin \zeta$, J is mapped onto

$$K : -1 \leq u \leq 1,\ v = 0$$

Hence, under the mapping $w = \cos z$, the interval I is mapped onto the interval K.

9.7 RIEMANN SURFACES

In this section, we propose to introduce the idea of Riemann surface, with illustrative examples.

Let $w = z^{1/3}$. This multi-valued function may be split into the following three parts:

$$(16)\qquad \begin{cases} w_0 = r^{1/3}\, e^{i\theta/3}, & 0 \le \theta < 2\pi \\ w_1 = r^{1/3}\, e^{i(\theta+2\pi)/3}, & 0 \le \theta < 2\pi \\ w_2 = r^{1/3}\, e^{i(\theta+4\pi)/3}, & 0 \le \theta < 2\pi \end{cases}$$

Note that each of these expressions defines a single-valued function that maps the entire z-plane onto "one third" of the w-plane. We can write this in the following form:

w_0 maps the z-plane onto the wedge
$0 \le \arg w \le 2\,\pi/3$ plus $w = 0$,
w_1 maps the z-plane onto the wedge
$2\,\pi/3 \le \arg w < 4\,\pi/3$ plus $w = 0$,
w_2 maps the z-plane onto the wedge
$4\,\pi/3 \le \arg w < 2\,\pi$ plus $w = 0$

Observe that all three of the functions above are analytic everywhere except along the non-negative real axis.

Now consider the functions defined by equations (16). We restrict their domain by deleting from it the negative real axis, and define

$$\psi_0\,(z) = r^{1/3}\, e^{i\theta/3},\ 0 < \theta < 2\pi,\ r \ne 0$$
$$\psi_1\,(z) = r^{1/3}\, e^{i\,(\theta+2\pi)/3},\ 0 < \theta < 2\pi,\ r \ne 0$$
$$\psi_2\,(z) = r^{1/3}\, e^{i\,(\theta+4\pi)/3},\ 0 < \theta < 2\pi,\ r \ne 0$$

Each of these functions is called a branch of the multi-valued function $w = z^{1/3}$. The ray consisting of the non-negative real axis is called a branch cut; and the point $z = 0$ from which the branch cut emanates is a branch point of each branch.

Write equations (16) in the following alternative form:

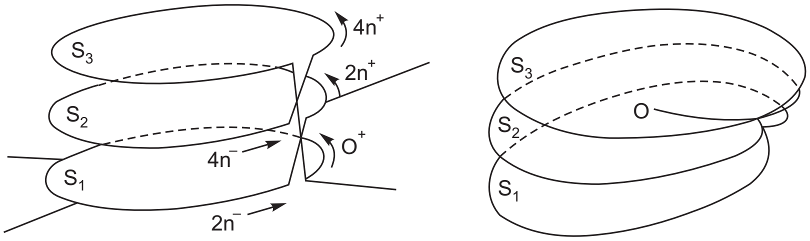
$$w_0 = r^{1/3}\, e^{i\theta/3},\ 0 \le \theta < 2\pi$$
$$w_1 = r^{1/3}\, e^{i\theta/3},\ 2\pi \le \theta < 4\pi$$
$$w_2 = r^{1/3}\, e^{i\theta/3},\ 4\pi \le \theta < 6\pi$$

It can be easily seen that the last three equations can be grouped together and written into one:

$$(17)\qquad w = r^{1/3}\, e^{i\theta/3},\ 0 \le \theta < 6\pi$$

A careful examination of these three alternatives will reveal that one fact remains unaltered in all of them: The z-plane must be traversed three times before the w-plane can be covered entirely. However, equation (17) lends itself to a "geometrical construction" which leads to a Riemann surface for the function $w = z^{1/3}$.

Instead of travelling the z-plane three times, we take three copies of the z-plane, S_1, S_2, S_3, forming a configuration that can be described as follows: [see Fig. 9.IX (*a*)]. As θ varies from 0 to 6π, trace the path of a point $z = re^{i\theta}$ as it describes a continuous curve around the origin, say, a circle. The point z starts from the positive real axis and travels on S_1 until its argument θ reaches 2π. At that instant, z ascends onto S_2 and it continues to travel around the origin while it remains on this second level. When its argument reaches 4π, z ascends onto S_3 and travels around the origin once again until θ reaches 6π. By this time, the function given by equation (17) has covered the w-plane completely, and the point z descends onto the first level S_1. The actual configuration can be visualized in the form of the sheets of the surface joined along the non-negative real axis [see Fig. 9.IX].



(a) (b)

Fig. 9.IX

This scheme has enables us to have a one-to-one mapping from the three-sheet Riemann surface onto the w-plane by means of the function given in equation (17). In a similar way one can visualize the Riemann surface of $w = z^{1/2}$ (with two sheets), of $w = z^{1/4}$ (with four sheets) and, in general, of $w = z^{1/n}$ (with n sheets).

Example: *Riemann Surface of the Natural Logarithm*

We recall that the multi-valued function

(18) $\quad w = \log z$

can be written as

$$w = \log |z| + i(\arg z + 2k\pi), \ (k = \text{integer})$$

Then, as k ranges over all integral values, we obtain the single-valued functions

(19) $\quad \ldots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \ldots .$

Note that each of these maps the z-plane; except $z = 0$, onto a horizontal strip of the w-plane, having width 2π. Thus, in order to cover the entire w-plane we map the z-plane an infinite number of times, by using all the functions given in (19). We now employ an infinite number of copies of the z-plane

$$\ldots, S_{-3}, S_{-2}, S_{-1}, S_0, S_1, S_2, S_3, \ldots .$$

each with its origin deleted and with a slit starting at the origin and running along the non-positive real axis. Consecutive sheets are then glued together along the slits so that they form a continuous sheet which resembles like a circular stair case of infinite height and depth. A point moving around the origin in a counterclockwise direction will be going up the staircase, and everytime it completes a circle (centred at the origin), its position will be on the next sheet up, directly above the starting point.

Observe that the above geometrical scheme enables us to think of the multivalued function $w = \log z$ as a single valued, mapping its Riemann surface in a one-to-one fashion onto the entire w-plane.

EXERCISES

1. Show that the mapping $w = z^2$ transforms every straight line into a parabola.
2. Consider the mapping $w = z^2$. Find the area of the image of the square $0 \leq x \leq 1$, $0 \leq y \leq 1$.

   Use $J = |f'|^2$ where J is the Jacobian of the mapping.
3. Find the image of Re $z > 0$ and Im $z > 0$ under the mapping $w = \log z$.
4. Find the image of the strip $-\pi < y \leq \pi$ under the mapping $w = 1 + e^z$.
5. Let $G = \{z : 0 < |z| < 1\}$. Find the map which maps G conformally onto the open unit disk.
6. Let G be a region and suppose that $f : G \to C$ is analytic such that $f(G)$ is a subset of a circle. Prove that f is constant.
7. Determine the fixed point, the rotation and dilation of the following mappings:
   - (*i*) $w = iz + 1$
   - (*ii*) $w = 2iz + 1 + i$
   - (*iii*) $w = (2 + 4i)\, z + 2$
8. Prove that if the complex numbers z_1, z_2, z_3 and z_4 lie on a circle, then their cross-ratio is real.
9. Determine the Möbius transformation which maps the following:
   - (*i*) the half plane Re $z \geq 0$ onto the half plane Im $z \geq 0$.
   - (*ii*) the upper half plane Im $z \geq 0$ onto itself.
   - (*iii*) the upper half plane Im $z \geq 0$ onto the lower half plane Im $z \leq 0$.
   - (*iv*) the half plane Im $z \geq 0$ onto the closed disk $\overline{D}$ (0, 1).
10. Suppose that T is a Möbius transformation with fixed points z_1 and z_2. Let S be a Möbius transformation. Prove that $S^{-1}\, TS$ has fixed points $S^{-1}\, z_1$ and $S^{-1}\, z_2$.
11. Prove that if the two fixed points z_1, z_2 of a Möbius transformation T coincide, then T is of the form

$$w = T(z) = \frac{K}{z - z_1} + z_2 \text{ where } K \in C.$$

12. (*a*) Show that a Möbius transformation T has 0 and ∞ as its only fixed points iff it is a dilation.

    (*b*) Show that a Möbius transformation T has ∞ as its only fixed point iff it is a translation.
13. Show that a Möbius transformation T satisfies $T(0) = \infty$ and $T(\infty) = 0$ iff $T_z = z_1 z^{-1}$ for some $z_1 \in C$.

14. Find the Möbius transformation which maps

(*i*) the points a, b, c into $0, 1, \infty$

(*ii*) the points $0, 1, \infty$ into a, b, c

(*iii*) the points $-1, 0, 2$ into $0, 3, 6$.

15. Find the Möbius transformation which maps

(*i*) $1, i, -1$ onto $i, -1, 1$

(*ii*) $i, -1, 1$ onto $-1, -i, 1$

(*iii*) $-1, -i, 1$ onto $-1, 0, 1$

(*iv*) $-1, 0, 1$ onto $-1, i, 1$

(*v*) $-1, i, 1$ onto $1, i, -1$

16. Find the Möbius transformation which maps

(*i*) $0, 1, \infty$ onto $1, \infty, 0$

(*ii*) $0, 1, \infty$ onto $1, -1, i$

(*iii*) $0, 1, \infty$ onto $-1, 0, 1$

(*iv*) $0, 1, \infty$ onto $-1, -i, 1$

17. Let $z_1, z_2, z_1 \neq z_2$, be the two fixed points of a Möbius transformation T. Show that T may be written as:

$$\frac{w - z_1}{w - z_2} = K\frac{z - z_1}{z - z_2}$$

Show that the cross-ratio of z_1, z_2, z, w is constant.

18. Let z_1, z_2, z_3, z_4 be distinct complex numbers. Suppose that they lie on the same circle, in that order. Prove that

$$|z_1 - z_3|\,|z_2 - z_4| = |z_1 - z_2|\,|z_3 - z_4| + |z_2 - z_3|\,|z_4 - z_1|$$

19. Recall: A function f with domain A and range B is given. We say that f is **univalent** if for $x_1, x_2 \in A$, the equation $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

We say that two regions B and B^* in C are **conformally equivalent** if there exists a **univalent function** $g : B \to B^*$ such that $B^* = g(B)$.

The mapping g is called a **conformal equivalence.** Prove that a conformal equivalence mapping C onto C is necessarily a polynomial.

20. With reference to problem (19), show that a conformal equivalence mapping C onto C is necessarily a nonconstant polynomial function.

10

HARMONIC FUNCTIONS

10.1 DEFINITIONS OF HARMONIC FUNCTIONS

Let f be a (real or complex valued) differentiable function of the real variables x and y. Consider the differential

(1) $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

The functions $z = x + iy$ and $\bar{z} = x - iy$ have differentials

(2) $$dz = dx + idy,\ d\bar{z} = dx - idy.$$

Thus

(3) $$dx = \frac{1}{2}(dz + d\bar{z}),\ dy = \frac{1}{2i}(dz - d\bar{z}).$$

Substituting this in (1) we have

(4) $$df = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)d\bar{z}$$

Write

(5) $$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

With this notation, we get the equation

(6) $$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}.$$

The condition

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

can now be written as

(7) $$\frac{\partial f}{\partial \bar{z}} = 0.$$

Definition: Let Ω be an open subset of C. A function $f(x, y)$ of two real variables x and y defined in an open set Ω is said to be **harmonic** in Ω if it has **continuous second partial derivatives** and satisfies the **Laplace's equation**

$$(8) \qquad \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Differentiating $\frac{\partial}{\partial \bar{z}}$ and $\frac{\partial}{\partial z}$ with respect to the complex variables $z = x + iy$ and $\bar{z} = x - iy$, we find that

$$(9) \qquad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\, \partial \bar{z}} \cdot$$

Thus the condition (8) is equivalent to

$$(10) \qquad \frac{\partial^2 f}{\partial z\, \partial \bar{z}} = 0.$$

Note that the condition (10) expresses that f is a harmonic function. Observe that by (8), a necessary and sufficient condition for a complex valued function $f = u + iv$ (u and v being real-valued) to be harmonic is that u and v are harmonic.

Definition: If $f : \Omega \to C$ is an analytic function, then $v = \operatorname{Im} f$ is the **conjugate harmonic function** of $u = \operatorname{Re} f$, and u is the conjugate harmonic function of $-v$.

10.2 HARMONIC FUNCTIONS AND ANALYTIC FUNCTIONS

Theorem 1. *Let $f : \Omega \to C$ be analytic. Then f is harmonic.*

Proof : Since f is analytic, it is infinitely differentiable, and by taking the derivative $\frac{\partial}{\partial z}$ of the relation

$$\frac{\partial f}{\partial \bar{z}} = 0, \text{ we get}$$

$$\frac{\partial^2 f}{\partial z\, \partial \bar{z}} = 0.$$

Hence f is harmonic.

Corollary 1 : The real and imaginary parts of an analytic function are harmonic functions.

Example: $\log |z|$ is harmonic in the whole plane C excluding the point $z = 0$. $\log z$ has a branch in some neighbourhood of each point $z \neq 0$ and $\log |z|$ is the real part of such a branch.

Theorem 2. *Let Ω be an open subset of C. Let $u(x, y)$ be a real harmonic function in Ω. Then in a neighbourhood of each point of Ω, u is the real part of an analytic function f which is determined upto addition of a constant.*

Proof : Since u is harmonic

$$\frac{\partial^2 u}{\partial z\, \partial \bar{z}} = 0.$$

Hence $\frac{\partial u}{\partial z}$ is analytic in Ω. The differential form $2\frac{\partial u}{\partial z}\, dz$ has therefore primitive f locally. In other words,

$$(11) \qquad df = 2\frac{\partial u}{\partial z} dz.$$

The relation (11) shows that f is analytic. Taking the complex conjugate of the relation (11) we obtain

$$(12) \qquad d\bar{f} = 2\frac{\partial u}{\partial \bar{z}} d\bar{z}.$$

Adding (11) and (12) we get

$$\frac{1}{2} d(f + \bar{f}) = du.$$

Thus u is equal to the real part of f with a real constant added if necessary.

It remains to show that if two analytic functions f_1 and f_2 in a neighbourhood of the same point have the same real part, then their difference $f = f_1 - f_2$ is constant. We have

$$d\,(f + \bar{f}) = 0$$

i.e.

$$\frac{\partial f}{\partial z} dz + \frac{\partial \bar{f}}{\partial \bar{z}} d\bar{z} = 0$$

which implies that $\frac{\partial f}{\partial z} = 0$ and $\frac{\partial \bar{f}}{\partial \bar{z}} = 0$. This completes the proof.

Note the Theorem 2 says only that any real harmonic function is locally the real part of an analytic function. Given a real harmonic function u in an open set Ω, there does not necessarily exist an analytic function f in the whole of Ω, whose real part is equal to u. For example, when $\Omega = C \sim \{0\}$, $\log |z|$ is not the real part of an analytic function in Ω.

Theorem 3. *Let $u : \Omega \to C$ be harmonic. Then u is infinitely differentiable.*

Proof : Let $z_0 = x_0 + iy_0$ be fixed in Ω. Choose δ such that $D\,(z_0; \delta) \subset \Omega$. Then there is a harmonic function v in $D\,(z_0; \delta)$ such that $f = u + iv$ is analytic in $D\,(z_0; \delta)$. Thus f is infinitely differentiable on $D\,(z_0; \delta)$ and hence u is infinitely differentiable.

The next result is analogous to Cauchy Integral Formula.

Theorem 4 (Mean Value Theorem). *Let $u : \Omega \to R$ be a harmonic function and let $\overline{D}\ (a; r) \subset \Omega$. If γ is the circle $|z - a| = r$, then*

$$(13) \qquad u\,(a) = \frac{1}{2\pi}\int_0^{2\pi} u\,(a + re^{i\theta})\, d\theta.$$

Proof : Choose a disk D such that $\overline{D}\ (a\ ; r) \subset D \subset \Omega$. Let f be analytic on D where $u = \operatorname{Re} f$. It follows from Cauchy's Integral formula that

$$f(a) = \frac{1}{2\pi i}\int_0^{2\pi} f\,(a + re^{i\theta})\, d\theta. \tag{14}$$

By equating the real part of (13) we get the result.

Theorem 5 (Maximum Principle). *Let u be a continuous real valued function on a region Ω satisfying the Mean Value Theorem. Let a ∈ Ω such that*

$$u\,(a) \geq u\,(z) \text{ for all } z \in \Omega,$$

then u is a constant function.

Proof : Define the set S by

$$S = \{z \in \Omega : u\,(z) = u\,(a)\}.$$

Since u is continuous, the set S is closed in Ω. Fix z_0 in S and choose r such that $\overline{D}\ (z_0 : r) \subset \Omega$. Let b be a point in $D\,(z_0; r)$ such that $u\,(a) \neq u\,(b)$, then $u\,(a) > u\,(b)$. Since u is continuous, we have $u\,(z) < u\,(a) = u\,(z_0)$ for all z in a neighbourhood of b. In particular, let $\rho = |\, z_0 - b\,|$ and $b = z_0 + \rho e^{i\alpha}$ where $0 \leq \alpha < 2\pi$. Then there exists a proper interval say I of $[0, 2\pi]$ such that $\alpha \in I$ and

$$u\,(z_0) > u\,(z_0 + \rho e^{i\beta}) \text{ for all } \beta \text{ in } I.$$

Hence, by Theorem 2

$$u\,(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\,(z_0 + \rho e^{i\beta})\, d\beta < u\,(z_0)$$

which is a contradiction. Hence $D\,(z_0; r) \subset S$ and S is open. Since Ω is connected, $S = \Omega$ and the proof is complete.

By considering the function $-\,u$ and appealing to Theorem 5 we get the **Minimum principle** theorem.

Theorem 6 (Minimum Principle). *Let u be a continuous real-valued function on a region Ω satisfying the Mean Value theorem. Let a ∈ Ω such that u (a) ≤ u (z) for all z ∈ Ω, then u is a constant function.*

10.3 HARMONIC FUNCTIONS ON A DISK

The maximum principle has the following consequences. If $u\,(z)$ is harmonic on a region Ω which contains a closed and bounded set S, then it is uniquely determined by its values on the boundary of S. If u_1 and u_2 are two harmonic functions with the same boundary values, then $u_1 - u_2$ is harmonic with the boundary value 0. By the Maximum and Minimum principle we find that $u_1 - u_2$ must be identically zero on S.

Now the problem arises of finding u when the boundary values are given.

This type of problem is known as the **Dirichlet problem** for the Laplace equation in two variables. the Dirichlet problem for a circular disk can be solved by

Poisson formula. We derive below the Poisson formula by using Cauchy's integral formula. Recall that the Cauchy's integral formula is given by

$$(15) \qquad f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

where γ is the circle represented by

$$\zeta = \mathrm{R}e^{i\phi} \; (0 \le \phi \le 2\pi)$$

and the function

$$f(z) = u(r, \theta) + iv(r, \theta) \; (z = re^{i\theta})$$

is analytic in a simply connected region Ω containing γ.

Since $\qquad d\zeta = i\mathrm{R}e^{i\phi}\, d\phi = i\zeta d\phi$

we have from (15)

$$(16) \qquad f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \frac{\zeta}{\zeta - z}\, d\phi .$$

Consider a point η outside γ where $\eta = \zeta\bar{\zeta} \,|\, \bar{z}$. Then the integral in (15) is analytic in the disk $|z| \le R$ and the integral is zero. Hence

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - \eta}\, d\zeta = 0$$

i.e.

$$\frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \frac{(\zeta)}{\zeta - \eta}\, d\phi = 0$$

i.e.

$$(17) \qquad \frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \frac{\bar{z}}{\bar{z} - \bar{\zeta}}\, d\phi = 0$$

Subtracting (17) from (16) and using the expression

$$(18) \qquad \frac{\zeta}{\zeta - z} - \frac{\bar{z}}{\bar{z} - \bar{\zeta}} = \frac{\zeta\bar{\zeta} - z\bar{z}}{(\zeta - z)(\bar{\zeta} - \bar{z})}$$

we obtain

$$(19) \qquad f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \frac{\zeta\bar{\zeta} - z\bar{z}}{(\zeta - z)(\bar{\zeta} - \bar{z})}\, d\phi.$$

By the polar representations of z and ζ we find that the quotient in the integrand is equal to

$$\frac{R^2 - r^2}{(\mathrm{R}e^{i\phi} - re^{-i\theta})(\mathrm{R}e^{-i\phi} - re^{-i\theta})} = \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

Hence equating the real parts of (19) we obtain **Poisson's integral formula**

$$(20) \qquad u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} u(R, \phi) \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, d\phi.$$

This formula represents a harmonic function in terms of its values given on the boundary circle of the disk. On this circle this function is equal to $u(R, \phi)$, except at points where $u(R, \phi)$ is not continuous. The proof is given in the next section.

From (20) we derive an important series of the function u in terms of simple harmonic functions. We have seen that the quotient in the integrand of (20) was derived from (18), and the right side of (18) is the real part of $\zeta + z/\zeta - z$. Using geometric series expansion we obtain

$$(21) \qquad \zeta + z/\zeta - z = \frac{1+\frac{z}{\zeta}}{1-\frac{z}{\zeta}} = 1 + 2\sum_{n=1}^{\infty}\left(\frac{z}{\zeta}\right)^n .$$

Since $z = re^{i\theta}$ and $\zeta = \text{Re}^{i\phi}$, we have

$$\operatorname{Re}\left(\frac{z}{\zeta}\right)^n = \operatorname{Re}\left\{\frac{r^n}{R^n}e^{in\theta}\, e^{-in\phi}\right\}$$

$$(22) \qquad = \left(\frac{r}{R}\right)^n (\cos n\theta \cos n\phi + \sin n\theta \sin n\phi).$$

From (21) and (22) we obtain

$$\operatorname{Re}\frac{\zeta+z}{\zeta-z} = 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n (\cos n\theta \cos n\phi + \sin n\theta \sin n\phi).$$

This expression is equal to the quotient in (20), and by inserting the series in (20) and integrating term by term we have

$$(23) \qquad u(r, \theta) = a_0 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n (a_n \cos n\theta + b_n \sin n\theta)$$

where
$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} u(R, \phi)\, d\phi;$$

$$(24) \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} u(R, \phi)\cos n\phi\, d\phi;$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} u(R, \phi)\sin n\phi\, d\phi \quad (n = 1, 2, ...).$$

Observe that for $r = R$, the series (23) becomes the **Fourier series** $u(R, \phi)$.

Example: Find the potential $u(r, \theta)$ in the unit disk ($r < 1$) where the boundary values are

$$u(1, \phi) = \begin{cases} -\phi/\pi \ (-\pi < \phi < 0) \\ \phi/\pi \ (0 < \phi < \pi). \end{cases}$$

Since $u(1, \phi)$ is even, $b_n = 0$, and from (24) we have

$$a_0 = \frac{1}{2} \text{ and}$$

$$a_n = \frac{1}{\pi}\left[-\int_{-\pi}^{0} \phi/\pi \cos n\phi\, d\phi + \int_0^{\pi}\frac{\phi}{\pi}\cos n\phi\, d\phi\right]$$

$$= \frac{2}{n^2\pi^2}(\cos n\pi - 1).$$

Hence

$$a_n = \begin{cases} -4/n^2\,\pi^2, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

and the potential is

$$u\,(r, \theta) = \frac{1}{2} - \frac{4}{\pi^2}\left[r \cos\theta + \frac{r^3}{3^2}\cos 3\theta + \ldots\right].$$

10.4 CONSTRUCTION OF HARMONIC FUNCTIONS ON A DISK

In this section, we shall construct harmonic function on the unit disk with prescribed boundary value. We shall introduce first the method of **Dirac sequences,** which we use. We shall be concerned with **periodic functions** of period 2π, so we make that assumption from the very beginning.

Definition: A **Dirac sequence** is a sequence of real valued functions $\{Q_n\}$ of real variable, periodic of period 2π, which satisfies the following conditions:

(*i*) $Q_n\,(x) \geq 0$ for each n and x;

(*ii*) $\{Q_n\}$ is continuous, and

$$\int_0^{2\pi} Q_n\,(t)\,dt = 1;$$

(*iii*) Let ε and δ be given. There exists n_0 such that if $n \geq n_0$, then

$$\int_{-\pi}^{-\delta} Q_n\,(t)\,dt + \int_{\delta}^{\pi} Q_n(t)\,dt < \varepsilon.$$

We illustrate **Dirac Family** $\{Q_n\}$ in Fig. 10.I. Note that the condition (*iii*) means that for sufficiently large n the area under the curve is concentrated near 0. Observe that $Q_n\,(-x) = Q_n\,(x)$.

Let f be a periodic function of period 2π. We define the **convolution** of f with Q_n as

$$Q_n{}^* f = \int_{-\pi}^{\pi} Q_n\,(x-t)\,f\,(t)\,dt = \int_{-\pi}^{\pi} f(x-t)\,Q_n\,(t)\,dt.$$

Theorem 7. *Let f be a continuous function, periodic of period 2π. Then the sequence of functions $\{Q_n * f\}$ converges uniformly to f.*

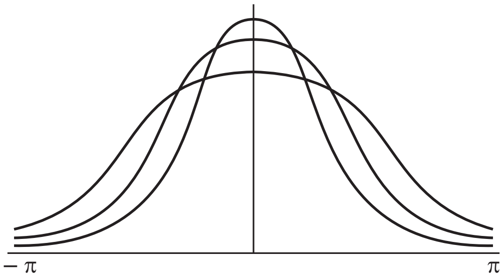


Fig. 10.I

Proof : Write

$$w_n(x) = Q_n{}^* f = \int_{-\pi}^{\pi} f(x-t)\, Q_n(t)\, dt.$$

If follows from condition (*ii*) that

$$f(x) = f(x) \int_{-\pi}^{\pi} Q_n(t)\, dt = \int_{-\pi}^{\pi} f(x)\, Q_n(t)\, dt.$$

Hence

$$(25) \qquad w_n(x) - f(x) = \int_{-\pi}^{\pi} (f(x-t) - f(x))\, Q_n(t)\, dt.$$

By the compactness of the circle, and the uniform continuity of *f*, it follows that

$$| f(x-t) - f(x) | < \varepsilon \text{ whenever } | t | < \delta.$$

Let $| f(x) | \leq M$. Then choose n_0 such that if $n \geq n_0$.

$$\int_{-\pi}^{-\delta} Q_n(t)\, dt + \int_{\delta}^{\pi} Q_n(t)\, dt < \frac{\varepsilon}{2M}.$$

Now

$$| w_n(x) - f(x) | \leq \int_{-\pi}^{-\infty} + \int_{-\delta}^{\delta} + \int_{\delta}^{\pi} | f(x-t) - f(x) |\, Q_n(t)\, dt$$

$$= I_1 + I_2 + I_3.$$

Since *M* is the bound for *f*, it follows that

$$| f(x-t) - f(x) | \leq 2M.$$

Thus,

$$I_1 + I_3 \leq 2M \left(\int_{-\pi}^{-\delta} Q_n(t)\, dt + \int_{\delta}^{\pi} Q_n(t)\, dt \right) < \varepsilon.$$

Also
$$I_2 = \int_{-\delta}^{\delta} | f(x) - f(x-t) | Q_n(t)\, dt$$

$$\leq \int_{-\delta}^{\delta} \varepsilon\, Q_n(t)\, dt \leq \int_{-\pi}^{\pi} \varepsilon\, Q_n(t)\, dt \leq \varepsilon.$$

This completes the proof of the theorem.

Observe that if Q_n has continuous derivatives, then

$$(26) \qquad \frac{d}{dx}(Q_n * f)(x) = \left(\frac{d}{dx} Q_n \right) * f.$$

It was convenient to introduce the general Dirac family $\{Q_n\}$, but we shall deal with families, indexed by *r* with $0 < r < 1$ and *r* approaching to 1. In this context, we need to define the **Poisson kernel** which is of basic importance.

The Poisson kernel is defined by

$$(27) \qquad P_r(\theta) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta}.$$

Let $\zeta = re^{i\theta}$, $0 \le r < 1$, then

$$\frac{1+re^{i\theta}}{1-re^{i\theta}} = (1+\zeta)(1+\zeta+\zeta^2+\ldots)$$

$$= 1 + 2\sum_{k=1}^{\infty}\zeta^k = 1 + 2\sum_{k=1}^{\infty} r^k e^{ik\theta}.$$

Thus $$\operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = 1 + 2\sum_{k=1}^{\infty} r^k \cos k\theta$$

$$= 1 + 2\sum_{k=1}^{\infty} r^k\left(\frac{e^{ik\theta}+e^{-ik\theta}}{2}\right)$$

$$= 2\pi P_r(\theta).$$

Observe that

$$\frac{1+re^{i\theta}}{1-re^{i\theta}} = \frac{1+r(e^{i\theta}-e^{-i\theta})-r^2}{|1-re^{i\theta}|^2},$$

$$\operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = \frac{1-r^2}{1-2r\cos\theta+r^2}.$$

Hence

$$P_r(\theta) = \frac{1}{2\pi}\frac{(1-r^2)}{(1-2r\cos\theta+r^2)}. \tag{28}$$

The Poisson kernel satisfies the following properties.

Properties: (*i*) $P_r(\theta) \ge 0$ for all r, θ;

(*ii*) $\int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1;$

(*iii*) Let ε and δ be given. There exists ρ, $0 < \rho < 1$, such that if $\rho < r < 1$, then

$$\int_{-\pi}^{-\delta} P_r(\theta)\, d\theta + \int_{\delta}^{\pi} P_r(\theta)\, d\theta < \varepsilon.$$

Proof: (*i*) By the equation (28),

$$P_r(\theta) = \frac{1-r^2}{2\pi}(|1-re^{i\theta}|)^{-2}, \text{ and since } r<1,\ P_r(\theta) \ge 0.$$

(*ii*) For a fixed r, $0 \le r < 1$, the series

$$\frac{1}{2\pi}\sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta}$$

converges uniformly in θ. Thus,

$$\int_{-\pi}^{\pi} P_r(\theta)\, d\theta = \sum_{k=-\infty}^{\infty} r^{|k|}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\, d\theta = 1.$$

By using (28) it can be easily verified that the property (*iii*) holds. The verification of (*iii*) is left to the reader.

Note that with these properties we view $P_r(\theta)$ not as a function of r and θ but a family of functions of θ, indexed by r. In other words, we view $\{P_r\}$ as a Dirac family, with $r \to 1$. Write

$$w_r = P_r * f,$$

where $$w_r(\theta) = w(r, \theta).$$

Then $w_r(\theta)$ is a function on the open unit disk. By theorem 7, $w_r(\theta)$ converges uniformly to $f(\theta)$ as r approaches 1.

The next theorem shows that the Dirichlet problem can be solved for the open disk $0 \le r < 1$ and $0 \le \theta \le 2\pi$.

Theorem 8. *The function* $(r, \theta) \to w_r(\theta)$ *is harmonic on the open disk* $0 \le r < 1$ *and* $0 \le \theta \le 2\pi$.

Proof: We recall that the **Laplace operator** in polar coordinates is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

By applying this operator to Poisson kernel $P_r(\theta)$, and differentiating the series $\frac{1}{2\pi}\sum_{k=-\infty}^{\infty} r^{|k|}e^{ik\theta}$ term by term, we find that

$$(29) \qquad \left[r^2\left(\frac{\partial^2}{\partial r^2}\right) + r\left(\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial \theta^2}\right] P_r(\theta) = 0.$$

That is,

$\Delta P = 0$, where P is a function of two variables,

$P(r, \theta) = P_r(\theta)$.

Using (26), we obtain

$$\Delta\,((P_r{}^* f)(\theta)) = (\Delta P_r(\theta)) * f = 0.$$

This completes the proof of the theorem.

Consider now the original periodic function f as a boundary value on the circle.

Then the function u is defined by **convolution** for $0 \le r < 1$, and by continuity for $r = 1$, and is given by

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, f(e^{it})\, dt.$$

Thus, the theorem shows that there exists a harmonic function u having the prescribed continuous boundary value f on the circle.

Note that the results of this section can be extended to disks of arbitrary radius R by means of the Poisson kernel

$$P_r(\theta) = \frac{1}{2\pi} \cdot \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2}, \quad 0 \le r < R.$$

In this case, one needs to prove:

(*a*) the statements about a Dirac family;

(*b*) the existence and uniqueness of a harmonic function with prescribed continuous boundary value on the closed disk of radius R. We leave it as an exercise.

In many applications, it may happen that the boundary value function f has a finite number of discontinuities. In that case, assuming that f is bounded, a similar analysis of proof will show that the theorem is true. We state the theorem and the verification is left to the reader.

Theorem 9. *Let f be a bounded real valued function, periodic of period* 2π. *Let f be continuous on a compact set E. Then the sequence of functions* $\{Q_n * f\}$ *converges uniformly on E to f. Further, the function u defined by*

$$u\,(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r\,(\theta - t)\, f\,(e^{it})\, dt$$

is harmonic on the open disk $0 \leq r < 1$ *and* $0 \leq \theta \leq 2\pi$.

10.5 SOME PHYSICAL APPLICATIONS OF CONFORMAL MAPPING

We have seen in this chapter that there are physical problems in which we need to solve Dirichlet problems with piecewise constant boundary values. In particular, these Dirichlet problems can be solved by using Fourier series. In general situations, conformal mapping is useful because a conformal map carries harmonic functions to harmonic functions. Consequently, it may be possible to solve a complicated problem by solving a simpler problem and then transforming the solution by conformal mapping. This process is used in solving some problems regarding fluid flow, electrostatics, and heat. In order to explain how it is applied, we need some physical terminology, which we present first for fluid flow.

Terminology of Fluid Flow

Consider a fluid that is flowing in a channel or lake at uniform depth, and in horizontal layers, so that a horizontal cross-section at any level represents the whole flow. We may then idealize the flow as two-dimensional and take it as the flow of a plane sheet of two-dimensional fluid in some region. We represent a velocity vector v by $v = pi + qj$ that specifies the velocity at each point. We assume that the fluid is incompressible and non-viscous. The term non-viscous means that there is no internal friction, and no tendency for the fluid to stick to the boundary of the region in which it is flowing. We also assume that the flow is irrotational (i.e. curl $v = 0$), which in physical language says that there are no local whirlpools in the fluid. Under these assumptions, the velocity vector v is the gradient of a potential ϕ, so that $p = \dfrac{\partial \phi}{\partial x}$ and $q = \dfrac{\partial \phi}{\partial y}$, where ϕ is harmonic. Then ϕ is called the velocity potential, and the curves

ϕ = constant are equipotentials. The conjugate harmonic function ψ is called the steam function, and its level curves ψ = constant are orthogonal to the equipotentials. Since, the derivative of ϕ along an equipotential is zero, the flow is directed along the curves of constant ψ, which are the streamlines. The integral $\int_A \left(\frac{\partial \phi}{\partial n}\right) ds$ along an arc A of an equipotential is proportional to the amount of fluid crossing the arc A per unit time.

The analytic function $f = \phi + i\psi$ is the complex potential, f' is the complex velocity, and $|f'(z)|$ is the speed of z. Since the function $f(z)$ is also analytic, it follows that $-\psi$ and ϕ are the velocity potential and stream function of another flow, whose streamlines are the equipotentials of the original flow.

If $f'(z_0) = 0$, the speed is zero at z_0, then z_0 is called a stagnation point. On the other hand, if $f'(z) \to \infty$ as $z \to z_0$. We may conclude that the speed must be very large near z_0.

If the flow is taking place in a region whose boundary contains a curve C across which fluid cannot flow, then the velocity vector at points of C must be tangent to C. Therefore a rigid boundary is always a streamline. It follows that we can have an interesting flow inside a simple closed curve C only if there are singular points on C, points at which fluid is being supplied or removed.

Note that for an unbounded region (for example, the strip between two parallel lines) different parts of the boundary can be different streamlines, with a singular point at ∞.

Example:

(*a*) Consider the function

$f(z) = \alpha z$ (α is real and positive) as representing a flow. Then its velocity potential is $\phi = \alpha x$ and its stream function is $\psi = \alpha y$. The streamlines, which are the curves of the family. $\psi = c$ are the horizontal lines $y = c/\alpha$. Observe that the flow described by f is parallel to the real axis. Its velocity is $V = \alpha$ [see Fig. 10.II(*a*)].

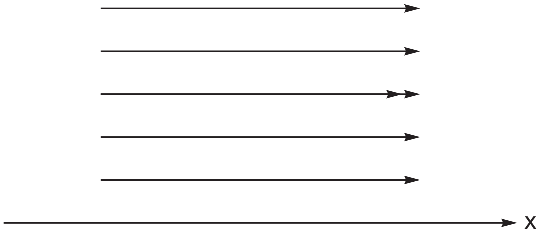


Fig. 10.II (*a*)

(*b*) We now examine a flow around 90° turn and for that we consider the function

$$w = z^{1/2}.$$

We know that, under this function, the upper half of the z-plane is mapped onto the first quadrant of the w-plane. Its inverse function

$$z = x + i\,y = (u^2 - v^2) + i\,(2uv)$$

transforms the first quadrant of the w-plane [Fig. 10.II (b)] into the upper half of the z-plane [Fig. 10.II (a)]. Then, it follows from the first part of this example that the velocity potential in this case is given by

$$\phi = \alpha x = \alpha\,(u^2 - v^2)$$

and the stream function becomes

$$\psi = \alpha y = 2\alpha\, uv.$$

Hence, the streamlines $\psi = c$ are the rectangular hyperbola

$$uv = \frac{c}{2\alpha}$$

[see Fig. 10.II(b)], and the complex potential is

$$f = w^2.$$

Finally, the velocity of the flow at any point (u, v) of the first quadrant is

$$V = \overline{f'} = 2\alpha\,(u - iv)$$

and hence its speed is

$$|\,V\,| = 2\alpha\,(u^2 + v^2)^{1/2}.$$

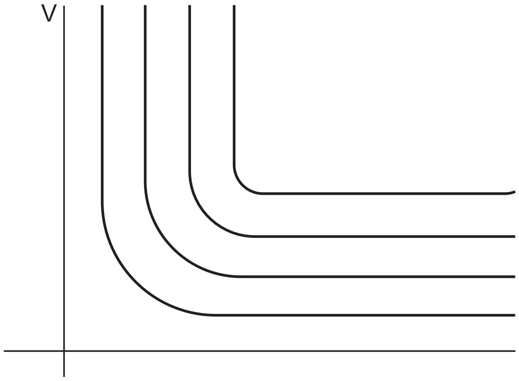


Fig. 10.II (b)

Some Special Flows

We shall first construct the flows generated by some simple analytic functions. We can then use conformal mapping to construct more complicated flows, using the principle that a conformal map transforms the solution of a Dirichlet problem to the solution of another Dirichlet problem.

Uniform Flows

Consider the simplest flow which has complex potential z. We could have taken the complex potential as cz, $c > 0$ but we can suppose that the units have been chosen so that $c = 1$. It follows that $\frac{dw}{dz} = 1$. Thus the velocity is constant, parallel to the real axis, and from left to right. If we take the real axis as rigid boundary, we have a uniform flow in the upper half plane. Suppose we take a line $y = k$ as a rigid boundary, then we have a flow in the channel between two parallel lines. In any of these cases the flow is referred to as a uniform flow. The streamlines are the lines

$$y = \text{constant}.$$

The following question arises: Where does the fluid come from, and where does it go ? We have to think of a source of fluid at ∞ and a sink, where fluid is being removed, also at ∞. It can be arranged in such a way that the fluid leaves the source, flows to the right, and then disappears at ∞. In order to give a mathematical description of sources and sinks, it will be easier to consider a source and a sink separately at finite points.

Sources and Sinks

We begin by considering the complex potential $w = m \log z$. Since, this is not a single valued function, we have to consider it only in the plane cut from 0 to ∞ (i.e. along the positive real axis). The complex velocity is

$$\frac{m}{z} = (m/r)\,(\cos\theta - i\sin\theta)$$

Thus the speed is m/r.

Since, the velocity vector is $(m/r)\,(i\cos\theta + j\sin\theta)$, it follows that the flow is outward along rays from the origin ($\theta = \text{constant}$). Hence, the cut along the positive real axis is along a streamline, and its presence is neglected. Now suppose that the fluid have unit density and let the fluid crosses a circle $|z| = r > 0$ in unit time.

This is given by

$$\int_{|z|=r} \frac{\partial\phi}{\partial n}\,ds = \int_0^{2\pi} \frac{\partial\,(m\log r)}{\partial r}\,r\,d\theta$$
$$= \int_0^{2\pi} m\,d\theta = 2\pi m,$$

so that the same amount of fluid crosses (in unit time) each circle centred at the origin. Hence, we say that $m \log z$ is the complex potential of a source at 0, of strength m.

A sink is defined as a negative source, its complex potential is

$$-\,m\log(z - z_0)$$

if it is located at a finite point z_0.

We illustrate the above facts in the form of a solved example.

Example: Let fluid emanates at a constant rate from an infinite line source perpendicular to the z-plane at $z = 0$.

(*a*) Show that the speed of the fluid at a distance r from the source is m/r where m is a constant.

(*b*) Show that the complex potential is $m \log z$.

(*c*) Consider portion of the line source of unit length as shown in Fig. 10.III. Suppose V_r is the radial velocity of the fluid at a distance r from the source. Let σ be the density of the fluid which is incompressible.

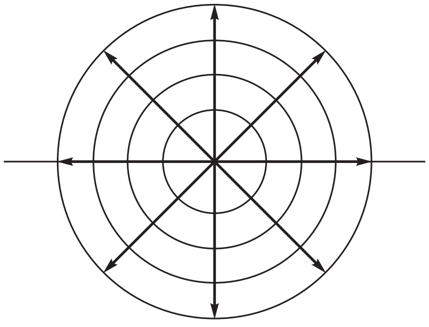


Fig. 10.III

We then have:

Mass of the fluid per unit time emanating from line source of unit length

= mass of fluid crossing surface of cylinder of radius r and height 1

= (Surface Area) · (Radial Velocity) · (Fluid Density)

$= (2\pi r \cdot 1)\,(V_r) \cdot \sigma = 2\pi r\, V_r\, \sigma.$

If this is taken as constant k, then

$$V_r = \frac{k}{2\pi\, r\, \sigma} = \frac{m}{r}$$

where $m = \dfrac{k}{2\pi\sigma}$ is called the strength of the source.

(*b*) Since $V_r = \dfrac{\partial\phi}{\partial r} = \dfrac{m}{r}$, it follows by integrating that

$$\phi = m \log r$$

where the constant of integration has been omitted. But $m \log r$ is the real part of $m \log z$ which is therefore the required complex potential.

10.6 SOME OTHER PHYSICAL INTERPRETATIONS

One can interpret every flow problem as a problem of the flow of heat or electrostatic potentials. The physical terminology is, of course, different. The heat problem can be considered as the problem of steady-state temperatures in a two dimensional lamina with its flat faces insulated. This problem of steady-state temperature can also be considered in a solid, homogeneous cylinder all of whose cross sections parallel to a given plane have the same shape (not necessarily circular), and with boundary conditions imposed on the edges of the lamina or the surface of the cylinder. A simple example of this model is the door of a refrigerator, with given distributions of temperature on the inside and outside surfaces. The temperature T satisfies Laplace's equation

$$\nabla^2 T = 0$$

The curves T = constant are called isothermals. Suppose that U is the harmonic function conjugate to T, then the curves U = constant are the lines along which heat flows. The quantity of heat crossing an arc A per unit time is proportional to

$$\int_A \left(\frac{\partial T}{\partial n}\right) ds.$$

When $\dfrac{\partial T}{\partial n} = 0$ on A, no heat is flowing through A and in this case we say that A is insulated.

We now solve one problem related to an insulated boundary.

Exercise: Prove that a conformal map transforms an insulated boundary to an insulated boundary.

Solution: Let $w = f(z)$ map a region Ω, bounded by a curve C, to a region Ω_1, bounded by a curve γ, in the w-plane, so that $w = u + iv$. Let $\phi\,(u, v)$ be harmonic in Ω_1. We shall prove that if $\dfrac{\partial\phi}{\partial n} = 0$ on an arc γ_1 of γ, then

$$\left(\frac{\partial}{\partial n}\right) \phi\,[u\,(x, y), v\,(x, y)] = 0$$

on the corresponding arc C_1 of C. The condition $\dfrac{\partial\phi}{\partial n} = 0$ implies that no heat flows across γ_1, and therefore γ_1 is an isothermal. We can identify this problem with a fluid flow problem considering γ_1 as a streamline and ϕ as velocity potential. Note that the conformal map transforms ϕ to a harmonic function and its harmonic conjugate ψ to a streamline. Thus, the conformal map transforms the orthogonal trajectory ϕ = constant to an equipotential where $\dfrac{\partial\phi}{\partial n} = 0$.

Observe that the value of $\dfrac{\partial\phi}{\partial n}$ is not preserved under a conformal map, but the condition $\dfrac{\partial\phi}{\partial n} = 0$ is preserved.

Electrostatic Problems

If a solid is perfect conductor, all charge is located on its surface. It follows that if the surface is represented by the simple closed curve C in the z-plane, the charges are in equilibrium on C and hence C is equipotential line. In other words, a conductor is the cross section of an actual conductor, and a point charge is the cross section of a uniform line charge perpendicular to the plane. Then the potential ϕ is harmonic in regions that contain no charges. Note that if $\phi + i\psi$ is an analytic function f, then the force on a unit charge is proportional to $|f'(z)|$. Curves ψ = constant are lines of force, and conductors are equipotentials, ϕ = constant.

Capacitor: Two conductors having charges of equal magnitude but of opposite sign, have a difference of potential which we denote v. The quantity ζ defined by

$$\eta = \zeta v$$

depends only on the geometry of the conductors. The conductors themselves form what is called a capacitor.

Illustration: The harmonic function in the unit disc with boundary values α on the upper semicircular boundary and β on the lower can be interpreted in various ways

(*a*) Describe a flow from a source at -1 to a sink at $+1$;

(*b*) Give the temperatures in the disc when the upper boundary is held at temperature α and the lower boundary at temperature β; and

(*c*) Describe the electric field in the disc when the upper boundary is at potential α, the lower boundary is at potential β, and there are bits of insulation at ± 1. Observe that in case (*c*) we have a capacitor.

EXERCISES

1. Denote by Δ

$$\Delta = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2$$

Prove that

$$\Delta = 4\,\delta\,\bar{\delta}$$

where $$\delta = \frac{\partial}{\partial z},\ \bar{\delta} = \frac{\partial}{\partial \bar{z}}.$$

2. Suppose that f is analytic and let $\bar{f} = u - iv$ be the complex conjugate of f. Show that

$$\delta\bar{f} = 0.$$

3. Suppose that $f : U \to V$ is an analytic isomorphism and let φ be a harmonic function on V, which is the real part of an analytic function.

Prove that $\varphi \circ f$ is harmonic.

4. Let f and g be two harmonic conjugate functions. Prove that

(*i*) $F = f^2 - g^2$ and $G = 2fg$

(*ii*) $\phi = e^f \cos g$ and $\psi = e^f \sin g$

are harmonic functions.

5. Suppose that U and V are conjugate harmonic functions and suppose that

$$\psi(x) = U^2(x, y) + V^2(x, y).$$

Prove that

$$\psi(x) = e^{ax+b} \text{ where } a \text{ and } b \text{ are real numbers.}$$

6. Suppose that U and V are conjugate harmonic functions and let

$$\phi(x) = \frac{V(x, y)}{U(x, y)}.$$

Prove that

$$\phi(x) = \tan(\alpha x + \beta) \text{ where } \alpha \text{ and } \beta \text{ are real numbers.}$$

7. Let $f(z) = \log z$.

Put $z = re^{i\theta}$. Then $f(z) = \log r + i\theta$, where $u = \log r$ and $v = \theta$. Draw the level curves u = constant and v = constant. Show that they intersect orthogonally.

8. Suppose that u is harmonic and bounded in $0 < |z| < r$. Prove that the origin is a removable singularity provided that $u(0)$ is defined properly and u is harmonic in $|z| < r$.

9. Suppose that $u(z)$ is harmonic in $0 < |z| < r$ and suppose $\lim\limits_{z \to 0} zu(z) = 0$. Prove that u is of the form

$$u(z) = \alpha \log |z| + v(z)$$

where α is a constant and v is harmonic in $|z| < r$.

10. Suppose that $U(\zeta)$ is piecewise continuous and bounded for all real ζ. Prove that

$$H_U(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-\zeta)^2 + y^2} U(\zeta)\, d\zeta$$

is harmonic in the upper half plane where $U(\zeta)$ represents the boundary values at points of continuity.

This is called Poisson's integral for the half plane.

11. Let $f(z)$ be harmonic and bounded in the upper half plane and let $f(z)$ be continuous on the real axis. Prove that $f(z)$ can be represented as a Poisson integral.

12. Let $P_y(t) = \dfrac{y}{\pi(t^2 + y^2)}$, $y > 0$. Prove that $\{P_y\}$ is a Dirac family.

13. Define

$$\phi(x, y) = P_y {}^* f(x) = \int_{-\infty}^{\infty} P_y(x - t)\, f(t)\, dt$$

where x and y are real and > 0. Prove that φ is harmonic. More precisely, prove that

$$\Delta\left(\frac{y}{(t-x)^2+y^2}\right) = 0, \Delta \text{ is the Laplace operator.}$$

14. **Harnack's theorem:** Suppose that $\{w_n\}$ is an increasing sequence of harmonic functions on the open disk D. Prove that either the sequence converges uniformly on compact subsets of the disk, or the sequence converges pointwise to ∞.

15. Suppose that $\{w_n\}$ is a sequence of harmonic functions on the open disk D. Suppose also that $\{w_n\}$ converges uniformly on compact subsets of the disk D. Prove that the limit of the sequence is harmonic.

16. Prove that a harmonic function satisfies the mean value property.

17. **Harnack inequality:**

Let $$k(\theta, z) = \text{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) = -\frac{1-|z|^2}{|e^{i\theta}-z|^2}.$$

Show that $$\frac{1-|z|}{1+|z|} \le k(\theta, z) \le \frac{1+|z|}{1-|z|}, |z| < 1.$$

18. Let $v(z) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} v(a+re^{i\theta})\, k\left(\theta, \frac{z-a}{r}\right) d\theta, |z-a| < r.$

Show that if $v(z) \ge 0, |z-a| \le r$, then

$$v(a)\,\frac{r-|z-a|}{r+|z-a|} \le v(z) \le v(a)\,\frac{r+|z-a|}{r-|z-a|}, |z-a| < r.$$

(**Hint:** Use Harnack inequality.)

19. Let G be a region in C and let $a \in G$. Let $\mathcal{F}$ denotes the set of functions φ harmonic on G such that $\varphi(a) = 1$, $\varphi(z) > 0$ for $z \in G$. Define $\lambda(z) = \inf\{\varphi(z) : \varphi \in \mathcal{F}\}$ and $\mu(z) = \sup\{\varphi(z) : \varphi \in \mathcal{F}\}$ for each $z \in G$. Prove that $0 < \lambda(z) \le 1 \le \mu(z) < \infty, z \in G$. Prove also that λ and μ are continuous on G.

(**Hint:** Consider the sets $\{\lambda(z) > 0\}$ and $\{\lambda(z) = 0\}$. By the aid of exercise 18 show that each set is open.)

20. **Subharmonic Functions:** Let Ω and Ω_1 be regions such that $\Omega \subseteq \Omega_1$. Suppose that $v(z)$ is a continuous real valued function defined on Ω. We say that the $v(z)$ is **subharmonic** in Ω if for any harmonic function $u(z)$ in Ω_1, the difference $v - u$ satisfies the maximum principle in Ω_1.

We say that a function is subharmonic at a point z_0 if it is subharmonic in a neighbourhood of z_0. Observe that if v is subharmonic in a neighbourhood of each point $z \in \Omega$, then it is subharmonic in Ω. Observe also that every harmonic function is subharmonic but the converse is not true.

Let Ω be a region and let $D_r(z_0)$ be an open disk centred at z_0 and radius r. Suppose that $v(z)$ is continuous.

Prove that $v(z)$ is subharmonic in Ω iff

$$v(z_0) \le \frac{1}{2\pi}\int_0^{2\pi} v(z_0+re^{i\theta})\, d\theta$$

holds for every $D_r(z_0) \subseteq \Omega$.

21. Using the results in the above exercise establish the following properties:

(*i*) if v is subharmonic, then Kv is also subharmonic, K is a constant and ≥ 0;

(*ii*) if v_1 and v_2 are subharmonic, then $v_1 + v_2$ is also subharmonic;

(*iii*) if v_1 and v_2 are subharmonic in Ω, then $v = \max(v_1, v_2)$ is also subharmonic in Ω.

22. Prove that the following functions are subharmonic:

(*i*) $|x|$ (*ii*) $|z|\alpha$, $(\alpha \geq 0)$ (*iii*) $\log(1 + |z|^2)$.

23. Let v be continuous and let v has continuous partial derivatives of second order. Prove that v is subharmonic iff $\Delta v \geq 0$.

11

WEIERSTRASS FACTORIZATION THEOREM

PART I

11.1 METRIC ON $\mathbb{C}$ (*G*, Ω)

Let G be an open set in C and let (Ω, d) be a complete metric space. We denote by $\mathbb{C}$ (G, Ω) the set of all continuous functions from G to Ω.

In order to introduce a metric on $\mathbb{C}$ (G, Ω), we first prove a theorem about open subsets of C.

Theorem 1. *Let G be open in C. Then there exists a sequence* $\{E_n\}$ *of compact subsets of G such that*

$$G = \bigcup_{n=1}^{\infty} E_n.$$

Further, the sets E_n *can be chosen to satisfy the following properties:*

(*i*) $E_n \subset \text{int. } E_{n+1}$.

(*ii*) $E \subset G$ *and E compact implies* $E \subset E_n$ *for some n.*

Proof : Define

$$E_n = \{z : |z| \le n\} \cap \left\{z : d(z, C - G) \ge \frac{1}{n}\right\}.$$

Since, E_n is the intersection of two closed subsets of C and E_n is bounded, it follows that E_n is compact. The set

$$S = \{z : |z| < n + 1\} \cap \left\{z : d(z, C - G) > \frac{1}{n+1}\right\}$$

is open and contains E_n. Also, S is contained in E_{n+1}. Thus the property (*i*) holds.

It can be easily verified that

$$G = \bigcup_{n=1}^{\infty} E_n \text{ and } G = \bigcup_{n=1}^{\infty} \text{int. } E_n.$$

If E is a compact subset of G, the sets $\{\text{int. } E_n\}$ form an open cover of E. Hence (*ii*) follows.

We now define the metric on $\mathbb{C}(G, \Omega)$. Suppose that $G = \bigcup_{n=1}^{\infty} E_n$ where each E_n is compact and $E_n \subset \text{int. } E_{n+1}$. Define

(1) $\rho_n(f, g) = \sup\{d(f(z), g(z)) : z \in E_n\}$

where f and g belong to $\mathbb{C}(G, \Omega)$. Also, define

(2) $$\rho(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f, g)}{1 + \rho_n(f, g)}.$$

Observe that the series in (2) is dominated by $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ and hence it converges.

Lemma 1: *Let (X, d) be a metric space and define*

$$\mu(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Then μ *is a metric on X.*

A set is open in (X, d) iff it is open in (X, μ*) ; a sequence is a Cauchy sequence in (X, d) iff it is a Cauchy sequence in (X,* μ*).*

The proof is left to the reader.

Theorem 2. $(\mathbb{C}(G, \Omega), \rho)$ *is a metric space.*

Proof : It can be easily seen that

$$\rho(f, g) = \rho(g, f).$$

Since each ρ_n satisfies the triangle inequality, it follows from the preceding lemma that ρ also satisfies the triangle inequality. Since $G = \bigcup_{n=1}^{\infty} E_n$, it follows that $f = g$ whenever $\rho(f, g) = 0$.

Theorem 3. *Define the metric* ρ *as in (2). Let* $\varepsilon > 0$ *be given. Then there is a* $\delta > 0$ *and a compact set* $E \subset G$ *such that for f and g in* $\mathbb{C}(G, \Omega)$

(3) $\sup\{d(f(z), g(z)) : z \in E\} < \delta$

$\Rightarrow \qquad \rho(f, g) < \varepsilon.$

Conversely, let $\delta > 0$ *and a compact set E be given. Then there is an* $\varepsilon > 0$ *such that for f and g in* $\mathbb{C}(G, \Omega)$,

(4) $\rho(f, g) < \varepsilon \Rightarrow \sup\{d(f(z), g(z)) : z \in E\} < \delta.$

Proof : Let $\varepsilon > 0$ be fixed and let m be a positive integer such that

$$\sum_{n=m+1}^{\infty} \left(\frac{1}{2}\right)^n < \frac{1}{2}\varepsilon$$

Put $E = E_m$. For suitably chosen $\delta > 0$ we have

$$\frac{t}{1+t} < \frac{1}{2}, \text{ where } 0 \le t < \delta.$$

By our hypothesis f and g are functions in $\mathbb{C}\,(G, \Omega)$ where

$$\sup\,\{d\,(f(z), g\,(z)) : z \in E\} < \delta.$$

Since $E_n \subset E_m$ for $1 \le n \le m$,

$$\rho_n\,(f, g) < \delta \quad \text{for } 1 \le n \le m.$$

Therefore,

$$\frac{\rho_n\,(f, g)}{1 + \rho_n\,(f, g)} < \frac{1}{2}\,\varepsilon \text{ for } 1 \le n \le m.$$

Hence

$$\rho\,(f, g) < \sum_{n=1}^{m} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\varepsilon\right) + \sum_{n=m+1}^{\infty} \left(\frac{1}{2}\right)^n < \varepsilon.$$

Conversely, let E and δ be given.

Since $$G = \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \text{int. } E_n$$

and E is compact, there is an integer $m \ge 1$ such that $E \subset E_m$.

From this we have

$$\rho_m\,(f, g) \ge \sup\,\{d\,f(z), g\,(z)) : z \in E\}.$$

Now choose $\varepsilon > 0$ such that

$$0 \le u < 2^m\,\varepsilon \;\Rightarrow\; \frac{u}{1-u} < \delta.$$

Hence $$\frac{t}{1+t} < 2^m\,\varepsilon \Rightarrow t < \delta.$$

Thus if $\rho\,(f, g) < \varepsilon$ then $\dfrac{\rho_m\,(f, g)}{1 + \rho_m\,(f, g)} < 2^m\,\varepsilon$ and this gives

$$\rho_m\,(f, g) < \delta.$$

Hence, the theorem is proved.

Lemma 2: *Let $\{f_n\}$ be a sequence in* $(\mathbb{C}\,(G, \Omega), \rho)$. *The sequence $\{f_n\}$ converges to f iff $\{f_n\}$ converges to f uniformly on all compact subsets of G.*

The proof is left to the reader.

Theorem 4. $\mathbb{C}\,(G, \Omega)$ *is a complete metric space.*

Proof: Suppose $\{f_n\}$ is a Cauchy-sequence in $\mathbb{C}\,(G, \Omega)$. Then the restrictions of the functions f_n to E give a Cauchy-sequence in $\mathbb{C}\,(E, \Omega)$ where E is compact and $E \subset G$. Hence $\{f_n\,(z)\}$ is a Cauchy-sequence in Ω. Thus there exists a point $f(z)$ in Ω such that $\lim\limits_{n \to \infty} f_z\,(z) = f(z)$.

We now got a function $f : G \to \Omega$ and have to prove that f is continuous and

$$\rho\,(f_n, f) \to 0$$

Let E be compact and fix $\delta > 0$. Choose N so that

$$\sup\,\{d\,(f_n\,(z), f_m\,(z)) : z \in E\} < \delta$$

holds for $n, m \geq N$. If z is arbitrary in E but fixed then there is an integer $m \geq N$ so that

$$d(f(z), f_m(z)) < \delta.$$

By triangle inequality, we get

$$d(f(z), f_n(z)) < 2\delta \text{ for all } n \geq N.$$

Since N does not depend on z we have

$$\sup \{d(f(z), f_n(z)) : z \in E\} \to 0$$

as $n \to \infty$. Hence $\{f_n\}$ converges uniformly on every compact set in G. Thus f is continuous. Finally, it follows from Lemma 2 that

$$\rho(f_n, f) \to 0.$$

11.2 SPACES OF ANALYTIC FUNCTIONS

The class of all analytic functions in G will be denoted by $H(G)$ where G is an open subset of C. In the following theorem we will prove that $H(G)$ is closed in $C(G, \mathbb{C})$.

Theorem 5. *Let $\{f_n\}$ be a sequence in $H(G)$. Suppose that f belongs to $\mathbb{C}(G, C)$ such that $f_n \to f$. Then f is analytic and*

$$f_n^k \to f^k \textit{ for each integer } k \geq 1.$$

Proof : Consider a disk $D \subset G$ and take a triangle T inside D. Since T is compact, $\{f_n\}$ converges uniformly over T.

Hence $$\lim \int_T f_n = \int_T f.$$

But since each f_n is analytic, we have

$$\lim \int_T f_n = \int_T f = 0.$$

Thus f is analytic in every disk $D \subset G$ and this gives that f is analytic in G.

To prove that $f_n^{(k)} \to f^{(k)}$, consider $D = \overline{D}(a: r) \subset G$. Then there is a number $r_1 > r$ such that $\overline{D}(a; r_1) \subset G$. Let $\gamma: |z - a| = r_1$ be a circle. Then by Cauchy's integral formula we have for $z \in D$,

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^{k+1}} \, d\zeta.$$

Hence

$$(5) \qquad |f_n^{(K)}(z) - f^{(K)}(z)| \leq \frac{k!}{2\pi} \frac{2\pi M_n r_1}{(r_1 - r)^{k+1}}$$

where $M_n = \sup \{|f_n(\zeta) - f(\zeta)| : |\zeta - a| = r_1\}$ and $|z - a| \leq r$.

Since $f_n \to f$, $\lim M_n = 0$. Hence, it follows from (5) that

$$f_n^{(K)} - f^{(K)} \text{ uniformly on } \overline{D}(a; r).$$

Now let E be an arbitrary compact subset of G and let $0 < r < d\,(E, \partial G)$. Then there are points $a_1, a_2, ..., a_n$ in E such that $E \subset \bigcup_{j=1}^{n} D\,(a_j\,;\,r)$. Since $f_n^{(K)} \to f^{(K)}$ uniformly on each $D\,(a_j\,;\,r)$, the convergence is uniform on E. Hence, the theorem is proved.

Observe that we have considered $H\,(G)$ as a subset of $\mathbb{C}\,(G, C)$ and the metric on $H\,(G)$ is the metric which it inherits from $\mathbb{C}\,(G, C)$.

Since $\mathbb{C}\,(G, C)$ is complete metric space we have the following results.

Corollary 1: $H\,(G)$ is a complete metric space.

Corollary 2: Let $f_n : G \to C$ be analytic. If $\sum_{n=1}^{\infty} f_n\,(z)$ converges uniformly on compact sets to $f\,(z)$ then

$$f^{(K)}\,(z) = \sum_{n=1}^{\infty} f_n^{(k)}\,(z).$$

Note that the above result has no analogue in the theory of functions of a real variable.

11.3 WEIERSTRASS FACTORIZATION THEOREM

The notion of convergence in $H\,(G)$ can be used to prove Weierstrass Factorization theorem. Before discussing the theorem we first define an **infinite product of complex numbers** and its convergence.

Infinite Products: An infinite product of complex numbers

$$z_1\, z_2 \,...\, z_n ... = \prod_{n=1}^{\infty} z_n \tag{6}$$

is obtained by taking the limit of the partial products

$$Z_n = z_1\, z_2 \,...\, z_n.$$

$\prod_{n=1}^{\infty} z_n$ is said to converge to the number z if $\lim_{n \to \infty} Z_n = z$ and is different from zero. Observe that if one of the numbers z_n is zero, then the limit is zero, and the convergence would not depend on the whole sequence of factors. Thus we define.

Definition: The infinite product (6) is said to converge iff at most a finite number of the factors are zero, and if the partial products formed by the non-vanishing factors tend to a finite limit which is different from zero.

We write $z_n = \dfrac{Z_n}{Z_{n-1}}$ and omit the zero factors. Thus in a convergent product the general factor z_n tends to 1. Hence we prefer to write all infinite products in the form

$$(7) \qquad \prod_{n=1}^{\infty} (1+z_n)$$

so that $z_n \to 0$ is a necessary condition for convergence.

Consider the power series expansion of log $(1 + z)$ about $z = 0$, *i.e.*

$$\log(1+z) = z - \frac{z^2}{2} + \dots .$$

It has radius of convergence 1. If $|z| < 1$ then

$$\left|1 - \frac{\log(1+z)}{z}\right| = \left|\frac{1}{2}z - \frac{1}{3}z^2 + \dots\right| \le \frac{1}{2}(|z| + |z|^2 + \dots)$$

$$= \frac{1}{2}\frac{|z|}{1-|z|} .$$

If $|z| < \frac{1}{2}$ then

$$\left|1 - \frac{|\log(1+z)|}{|z|}\right| \le \frac{1}{2}.$$

Thus we have for $|z| < \frac{1}{2}$

$$(8) \quad \frac{1}{2}|z| \le |\log(1+z)| \le \frac{3}{2}|z|.$$

An infinite product is said to be absolutely convergent iff the corresponding series $\sum_{n=1}^{\infty} \log(1+z_n)$ converges absolutely.

Theorem 6. *Let (X, d) be a compact metric space. Suppose* $\{u_n\}$ *is a sequence of continuous functions from X into C, such that* $\sum_{n=1}^{\infty} |u_x(x)|$ *converges uniformly on X. Then the product*

$$f(x) = \prod_{n=1}^{\infty} (1+u_n(x))$$

converges absolutely and uniformly on X. Also $f(x_0) = 0$ *at some* $x_0 \in X$ *iff* $u_n(x_0) = -1$ *for some n.*

Proof: The hypothesis implies that there is an integer n_0 such that $|u_n(x)| < \frac{1}{2}$ for all x in X and $n > n_0$. Hence Re $[1 + u_n(x)] > 0$. It now follows from inequality (8) that

$$|\log(1+u_n(x))| \le \frac{3}{2}|u_n(x)|$$

for all $n > n_0$ and $x \in X$.

Thus

$$v(x) = \sum_{n=n_0+1}^{\infty} \log(1+u_n(x))$$

converges uniformly on X. Since v is continuous and X is compact, v must be bounded. Hence

$$\exp v(x) = \prod_{n=n_0+1}^{\infty} (1+u_n(x))$$

converges uniformly on X (why ?).

Finally,

$$f(x) = [1+u_1(x)] \ldots [1+u_{n_0}(x)] \exp v(x)$$

and $\exp v(x) \neq 0$ for any $x \in X$. So $f(x_0) = 0$ at some $x_0 \in X$ iff $u_n(x_0) = -1$ for some n.

Theorem 7. *Let G be a region in C. Suppose $f_n \in H(G)$ such that no f_n is identically zero. If $\sum_{n=1}^{\infty} |1-f_n(z)|$ converges uniformly on compact subsets of G, then the product $\prod_{n=1}^{\infty} f_n(z)$ converges in $H(G)$ to an analytic function $f(z)$.*

Furthermore, if z_0 is a zero of f then z_0 is a zero of only a finite number of the functions f_n, and the multiplicity of the zero of f at z_0 is the sum of the multiplicities of the zeros of the functions f_n at z_0.

Proof : The first part follows from Theorem 6. For the second part, observe that $\overline{D}(z_0, r) \subset G$ and since $\sum_{n=1}^{\infty} (1-f_n(z))$ converges uniformly on $\overline{D}(z_0; r)$, it follows from Theorem 6 that

$$f(z) = \{f_1(z) \ldots f_n(z)\}\, u(z)$$

where u does not vanish in $\overline{D}(z_0; r)$. It can be checked easily that the multiplicity of the zero of f at z_0 is the sum of the multiplicities of the zeros of the functions f_n at z_0.

We now define the functions which were introduced by **Weierstrass.**

Definition: Set $E_0(z) = 1-z$, and for $p = 1, 2, 3, \ldots$

$$E_p(z) = (1-z) \exp\left\{z + \frac{z^2}{2} + \ldots + \frac{z^p}{p}\right\}.$$

These functions are called **elementary factors.** Note that $E_p\left(\frac{z}{z_0}\right)$ has a simple zero at $z = z_0$ and no other zero.

Lemma 3: *If $|z| \leq 1$ and $p \geq 0$, then*

$$|1 - E_p(z)| \leq |z|^{p+1}.$$

Proof : For $p = 0$, this is trivially true. For $p \geq 1$, let

$$E_p(z) = 1 + \sum_{k=1}^{\infty} a_k z^k$$

where p is fixed

$$E'_p(z) = \sum_{k=1}^{\infty} k\, a_k z^{k-1}$$

i.e. $$-E'_p(z) = z^p \exp\left\{z + \frac{z^2}{2} + \ldots + \frac{z^p}{p}\right\}.$$

So $-E'_p(z)$ has a zero of order p at $z = 0$, and the coefficients in the expansion of $-E'_p(z)$ in powers of z are all positive.

Hence, $a_k \leq 0$ for $k \geq p + 1$.

Thus

$$|a_K| = -a_K \text{ for } k \geq p + 1$$

and from this we have

$$\sum_{k=p+1}^{\infty} |a_k| = -\sum_{k=p+1}^{\infty} a_k = 1.$$

Hence, for $|z| \leq 1$

$$|E_p(z) - 1| = \left|\sum_{k=p+1}^{\infty} a_k z^k\right|$$

$$= |z|^{p+1} \left|\sum_{k=p+1}^{\infty} a_k z^{k-p-1}\right|$$

$$\leq |z|^{p+1} \sum_{k=p+1}^{\infty} |a_K| = |z|^{p+1}.$$

Thus the Lemma is proved.

Theorem 8. *Let $\{z_n\}$ be a sequence in C such that $z_n \neq 0$ and $|z_n| \to \infty$ as $n \to \infty$. If p_n is a sequence of non-negative integers such that*

$$(9) \quad \sum_{n=1}^{\infty} \left(\frac{r}{|z_n|}\right)^{p_n+1} < \infty$$

for all $r > 0$, then the infinite product

$$(10) \quad f(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$$

converges in H (C). The function f is an entire function which has a zero at each point z_n and which has no other zero in C. More precisely if z_0 occurs in the sequence $\{z_n\}$, exactly m times then f has a zero at $z = z_0$ of multiplicity m.

Furthermore, (9) is always satisfied if $p_n = n - 1$.

Proof : Suppose that there are integers p_n such that (9) is satisfied. Then, by Lemma 3

$$|1 - E_{p_n}\left(\frac{z}{z_n}\right)| \leq |\frac{z}{z_n}|^{p_n+1} \leq \left(\frac{r}{|z_n|}\right)^{p_n+1}$$

provided $|z| \leq r$ and $|z_n| \geq r$.

Since $\lim |z_n| = \infty$, for fixed $r > 0$ there is an integer N such that $|z_n| \geq r$ for all $n \geq N$. It now follows from (9) that the series

$$\sum_{n=1}^{\infty} |1 - E_{p_n}\left(\frac{z}{z_n}\right)|$$

converges uniformly on compact sets in C.

By Theorem 7, the infinite product $\prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$ converges in $H(C)$. Observe that $\{p_n\}$ can be found so that (9) holds for all r. For any r, $|z_n| > 2r$ for all $n \geq N$. Hence $\left(\frac{r}{|z_n|}\right) < \frac{1}{2}$ for all $n \geq N$. Thus if $p_n = n - 1$ for all n, the tail end of the series (9) is dominated by $\Sigma\, (1/2)^n$. Hence (9) converges.

There is an advantage in taking the constant p_n as small as possible. The resulting function $f(z)$ in (10) is then called the **canonical product** corresponding to $\{z_n\}$. For instance, choose $p_n = 0$, if $\sum_{n=1}^{\infty} \frac{1}{|z_n|} < \infty$, and the canonical product is

$$\prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right).$$

We now state the **Weierstrass Factorization theorem.**

Theorem 9. *Let F be an entire function and suppose that* $F(0) \neq 0$. *Let* $z_1, \ldots, z_n \ldots$ *be the zeros of f, listed according to their multiplicities. Then there is an entire function g and a sequence of non-negative integers* $\{p_n\}$ *such that*

$$(11) \quad F(z) = e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right).$$

Proof : Let f be the product in Theorem 8, formed with the zeros of F. Then F/f has removable singularities in C and hence F/f is an entire function. Further, F/f has no zero and since C is simply connected, there is an entire function g such that

$$F(z)/f(z) = e^{g(z)}$$

The result now follows.

Note that if F has a zero of order m at $z = 0$, the theorem holds for $F(z)/z^m$.

We now give an application of the Weierstrass Factorization Theorem to $\sin \pi z$.

The zeros of $\sin \pi z = \dfrac{1}{2i}(e^{i\pi z} - e^{-i\pi z})$ are the integers $z = \pm n$. Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges and $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ converges, we choose $p_n = 1$ and obtain a representation of the form

$$(12) \quad \sin \pi z = [\exp g(z)] \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}.$$

Taking the logarithmic derivatives of both sides of (12) we have

$$(13) \quad \pi \cot \pi z = \frac{1}{z} + g'(z) + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n}\right).$$

The convergence is uniform over compact subsets of C which does not contain the points $z = n$. Also

$$(14) \quad \pi \cot \pi z = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n}\right).$$

Comparing, we conclude that $g'(z) = 0$. Hence g is constant. Since $\lim\limits_{z \to 0} \dfrac{\sin \pi z}{z} = \pi$, we have $\exp(g(z)) = \pi$. Thus

$$\sin \pi z = \pi z \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}.$$

The terms of the infinite product corresponding to n and $-n$ can be arranged and we obtain

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

The convergence is uniform over compact subsets of the plane.

We now discuss an infinite series expansion, due to **Mittag-Leffler,** which is related to the Weierstrass product. The main idea is to represent a meromorphic function $f(z)$ by a series, each term of which contains the principal part of f at one of the singularities. Observe that the familiar partial fraction expansion for a rational function is an example of a Mittag-Leffler expansion, just as the factorization theorem for polynomials is an example of the Weierstrass expansion.

Suppose it is required to construct a function with simple poles of residue 1 at $z = 1, 2, 3, \ldots$. One would like to represent such a function by

$$\sum_{k=1}^{\infty} \frac{1}{z-k}.$$

This representation is not satisfactory, because the series diverges for every value of z. We represent such function by

$$\sum_{k=1}^{\infty}\left(\frac{1}{z-k}+\frac{1}{k}\right).$$

Note that the series converges except at points where the denominator vanishes.

We define the Mittag-Leffler primary terms for $n = 1, 2, \ldots$ by

$$\text{(15)} \quad L(w, n) = \frac{1}{w-1} + 1 + w + w^2 + \ldots + w^{n-1}$$

and

$$L(w, 0) = \frac{1}{w-1}$$

It follows that

$$L(w, n) = \frac{w^n}{w-1}$$

and

$$\text{(16)} \quad |L(w, n)| \leq 2|w|^n, \; |w| \leq \frac{1}{2}$$

Write, for $n = 1, 2, 3, \ldots,$

$$\text{(17)} \quad E(w, n) = (1-w)\exp\left(w + \frac{w^2}{2} + \ldots + \frac{w^n}{n}\right)$$

and

$$E(w, 0) = 1 - w.$$

Observe that there is a close relation between $L(w, n)$ and $E(w, n)$. Since, the first term of (15) is $\frac{1}{w-1}$, we find that

$$\int_0^w L(\zeta, n)\, d\zeta = \log E(w, n), |w| < 1.$$

Note that the path of integration is taken along the radius from 0 to w. Using (16) we get the estimate of the integral

$$\text{(18)} \quad |\log E(w, n)| \leq \frac{2|w|^{n+1}}{n+1}, |w| \leq \frac{1}{2}.$$

This is the basic inequality of Weierstrass expansion theory.

Theorem 10 (Mittag-Leffler). *Let* $\{\alpha_k\}$ *be sequence of distinct complex numbers such that* $|\alpha_k| \to \infty$. *Let* $\{\beta_k\}$ *be another sequence of complex numbers such that*

$$(19)\qquad \sum_{k=1}^{\infty} \frac{|\beta_k|}{|\alpha_k|^{n+1}} < \infty.$$

Then there exists a meromorphic function f whose only finite singularities are simple poles at α_k *with residue* β_k.

The function is represented by

$$(20)\qquad f(z) = \sum_{k=1}^{\infty} \frac{\beta_k}{\alpha_k} L\left(\frac{z}{\alpha_k}, n\right).$$

Proof: Let $r > 0$. Choose N so that $|\alpha_k| > 2r$ for $k > N$. Then for $|z| < r$ and $k > N$, $\left|\frac{z}{\alpha_k}\right| < \frac{1}{2}$. Hence, it follows from (16) that

$$\left|\frac{\beta_k}{\alpha_k} L\left(\frac{z}{\alpha_k}, n\right)\right| \le \frac{2|\beta_k| r^n}{|\alpha_k|^{n+1}}.$$

It now follows from (19) that the series

$$\sum_{k=N}^{\infty} \frac{\beta_k}{\alpha_k} L\left(\frac{z}{\alpha_k}, n\right)$$

converges uniformly in $|z| < r$.

Observe that the remaining terms of the series for $f(z)$ differ only by a polynomial form

$$\sum_{k=1}^{N-1} \frac{\beta_k}{\alpha_k} \frac{1}{z/\alpha_k - 1} = \sum_{k=1}^{N-1} \frac{\beta_k}{z - \alpha_k}$$

and hence have simple poles at $z = \alpha_k$ with residue β_k. This completes the proof of the theorem.

The above theorem is not the most general form of the Mittag-Leffler theorem. In its general form, the theorem states that a meromorphic function $f(z)$ can be constructed whose only poles are at $\{\alpha_k\}$ with the prescribed principal part. The proof of the general theorem is similar, in principle, to the proof of Theorem 10. The proof is given here for the sake of completeness.

Theorem 11. *Let* $\{\alpha_k\}$ *be a sequence of distinct complex numbers such that* $|\alpha_k| \to \infty$. *Let* $\{P_k\}$ *be polynomials without constant term. Then there exists a meromorphic function f whose only poles are at* $\{\alpha_k\}$ *with the prescribed principal part*

$$P_k\left(\frac{1}{z - \alpha_k}\right).$$

The function f is represented by

$$f(z) = \sum_k \left[P_k\left(\frac{1}{z-\alpha_k}\right) - Q_k(z) \right] + \psi(z)$$

where Q_k is some polynomial, and ψ is an entire function. The series converges absolutely and uniformly on any compact set not containing the poles at $\{\alpha_k\}$.

Proof: We assume without loss of generality that $\alpha_k \neq 0$ for all k. We represent

$$P_k\left(\frac{1}{z-\alpha_k}\right)$$

in a power series at the origin. Choose ρ_k such that $Q_k(z)$ is the sum of the terms of the degree ρ_k^{-1} in this series. It follows that Q_k is the polynomial of degree ρ_k. For $|z| \leq \frac{|\alpha_k|}{2}$ we find that

$$\left| P_k\left(\frac{1}{z-\alpha_k}\right) - Q_k(z) \right| \leq A_k \left| \frac{z}{\alpha_k} \right|^{\rho_k}$$

where A_k is some fixed number depending on $P_k\left(\frac{1}{z-\alpha_k}\right)$. Choose ρ_k sufficiently large so that

$$\frac{(A_k)^{1/\rho_k}}{|\alpha_k|} \to 0 \text{ as } |\alpha_k| \to \infty$$

and also such that the ρ_k form an increasing sequence $\rho_1 < \rho_2 < \ldots$. Write

$$(21) \qquad \sum_k \left[P_k\left(\frac{1}{z-\alpha_k}\right) - Q_k(z) \right]$$

$$= \sum_{k=1}^{N} \left[P_k\left(\frac{1}{z-\alpha_k}\right) - Q_k(z) \right] + \sum_{n=1}^{\infty} \left[P_k\left(\frac{1}{z-\alpha_k}\right) - Q_k(z) \right].$$

Given a radius r, let $|\alpha_N| \geq r$. Note that the first sum on the right of (21) is a finite sum. The second sum on the right of (21) is dominated by

$$(22) \qquad \sum_k \left(\frac{A_k}{|\alpha_k|^{\rho_k}} \right) z^{\rho_k}$$

provided $\qquad |z| \leq \frac{r}{2}.$

By taking the ρ_k-th root of the coefficients, we find that the radius of convergence is equal to ∞. Thus the series

$$\sum_k \left[P_k \left(\frac{1}{z-\alpha_k} \right) - Q_k\,(z) \right]$$

converges absolutely and uniformly for z in any compact set not containing the poles at $\{\alpha_k\}$.

Observe that the finite sum on the right of (21) has the desired poles, and the infinite sum on the right of (21) has no poles for $|\,z\,| < \frac{r}{2}$. Since, this is true for every r, this completes the proof of the theorem.

Example: Show that

$$\cot z = \frac{1}{z} + \sum_k \left(\frac{1}{z-k\pi} + \frac{1}{k\pi} \right)$$

where the summation extends over $k = \pm\,1, \pm\,2, \pm\,3, \ldots$.

Consider the function

$$\cot z - \frac{1}{z} = \frac{z\cos z - \sin z}{z \sin z}$$

Note that the function $\cot z - \frac{1}{z}$ has simple poles at $z = k\pi$, $k = \pm\,1, \pm\,2, \pm\,3, \ldots$ and the residue at these poles is

$$\lim_{z \to k\pi} (z - k\pi) \left(\frac{z\cos z - \sin z}{z \sin z} \right) = 1.$$

Since $$\lim_{z \to 0} \left(\cot z - \frac{1}{z} \right) = 0,$$

it follows that $\cot z - \frac{1}{z}$ has removable singularity at $z = 0$.

It can be easily checked that $\cot z - \frac{1}{z}$ is bounded on circles γ_N having centre at the origin and radius $r_N = \left(N + \frac{1}{2} \right) \pi$. Hence, it follows from Mittag-Leffler theorem that

$$\cot z - \frac{1}{z} = \sum_k \left(\frac{1}{z-k\pi} + \frac{1}{k\pi} \right).$$

EXERCISES

1. Prove that $\pi\,(1 + z_n)$ converges absolutely iff $\pi(1 + |\,z_n\,|)$ converges.
2. Let $0 < |\,\alpha\,| < 1$ and let $|\,z\,| \le r < 1$; prove that

$$\frac{\alpha + |\alpha|\,z}{(1+\bar{\alpha}\,z)\,\alpha} \le \frac{1+r}{1-r}$$

3. **(Blaschke products).** Let $\{\alpha_n\}$ be a sequence of complex numbers with $0 < |\alpha_n| < 1$ and let $\Sigma\,(1 - |\alpha_n|) < \infty$. Prove that

$$B(z) = \prod_{n=1}^{\infty} \frac{|\alpha_n|}{\alpha_n}\left(\frac{\alpha_n - z}{1 - \bar{\alpha}_n z}\right)$$

converges in $H(D(0, 1))$ and that $|B(z)| \le 1$.

4. Let $\alpha_n = 1 - n^{-2}$ and let B be the Blaschke product with zeros at these points α. Prove that $\lim\limits_{r \to 1} B(r) = 0$ if $0 < r < 1$. In fact, prove the estimate.

$$|B(r)| < \prod_{n=1}^{N-1} \frac{r - \alpha_n}{1 - \alpha_n r} < \prod_{n=1}^{N-1} \frac{\alpha_N - \alpha_n}{1 - \alpha_n} < e^{-N/2}$$

where $\alpha_{N-1} < r < \alpha_N$.

5. Find a sequence $\{\alpha_n\}$ in $D(0, 1)$ such that $\Sigma\,(1 - |\alpha_n|) < \infty$ and every number $e^{i\theta}$ is a limit point of $\{\alpha_n\}$.

6. Discuss the convergence of the following:

$$(i)\ \prod_{n=1}^{\infty} \frac{1}{n^{\rho}},\ (\rho > 0) \qquad (ii)\ \prod_{n=1}^{\infty}\left(1 + \frac{i}{n}\right) \qquad (iii)\ \prod_{n=2}^{\infty}\left(1 - \frac{i}{n^2}\right).$$

7. Suppose $\{f_n\}$ is a sequence in $H(G)$ where G is an open set. Let $\prod\limits_{n=1}^{\infty} f_n(z)$ converges in $H(G)$ to $f(z)$. Prove that

$$\sum_{k=1}^{\infty}\left[f_k(z) \prod_{n \ne k} f_n(z)\right]$$

converges in $H(G)$ to $f'(z)$.

8. Prove that
$$\prod_{n=1}^{\infty}\left(1 + \frac{z}{n}\right) e^{-z/n}$$

converges absolutely and uniformly on every compact set.

9. Prove that
$$f(z) = \prod_{n=1}^{\infty} (1 + q^{2n-1} e^{z})(1 + q^{2n-1} e^{-z})$$

where $|q| < 1$ is analytic in the whole complex plane and satisfies

$$f(z + 2 \log q) = q^{-1} e^{-z} f(z).$$

10. Find all entire functions f such that $|f(z)| = 1$ whenever $|z| = 1$.

11. Suppose that f is an entire function, and n is a positive integer. Prove that there is an entire function g such that $g^n = f$ iff the orders of the zeros of f are divisible by n.

12. Prove that there is a bounded analytic function f on $D(0, 1)$ such that each point of the unit circle is a singularity.

13. Let $z_n \to \infty$ and let β_n be arbitrary complex numbers. Prove that there exists an entire function $f(z)$ such that $f(z_n) = \beta_n$.

14. **(The Gamma function).** $\Gamma(z)$ is the meromorphic function on C with simple poles at $z = 0, -1, \ldots$ defined by

$$\Gamma(z) = e^{-\gamma z/n} \prod_{n=1}^{\infty}\left(1 + \frac{z}{n}\right)^{-1} e^{z/n},$$

where γ is constant chosen so that $\Gamma(1) = 1$. Show that $0 < \gamma < 1$.

15. Show that $\Gamma(z)\,\Gamma(1-z) = \pi \operatorname{cosec} \pi z$ where $z \neq$ an integer.

16. Show that

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right)$$

17. Prove that

$$\sin \pi(z+p) = e^{\pi z \cot \pi p} \prod_{-\infty}^{\infty}\left(1+\frac{z}{z+p}\right) e^{-z/n+p}.$$

where $p \neq$ an integer.

18. Prove that the residue of $\Gamma(z)$ at $-p$ is $(-1)^p/p!$

19. Prove that

$$\Gamma'/\Gamma(z) = \int_0^\infty \left[\frac{e^{-1}}{t} - \frac{e^{-zt}}{1-e^{-1}}\right] dt$$

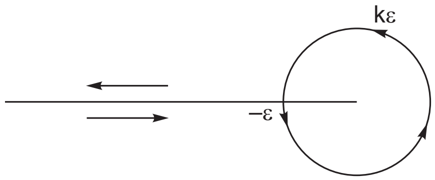
$$\left[\text{Use } \frac{1}{z+m} = \int_0^\infty \exp[-t(z+m)]\, dt\right]$$

20. **(Hankel's Integral)** Let

$$G(z) = \frac{e^{-z}}{1-e^{-z}} \text{ amd } F(z) = \frac{ze^z}{e^z-1}$$

Let $$H(s) = \int_C \frac{1}{z} F(z)\, z^2 \frac{dz}{z}$$

where the contour C is given by



Exercise 11.I

(*i*) Prove that H is an entire function

(*ii*) Prove by putting $z = -w$ and letting $\varepsilon \to 0$ that

$$H(s) = -(e^{\pi is} - e^{-\pi is}) \int_0^\infty G(t)\, t^2 \frac{dt}{t}$$

21. Let $\zeta(s) = \Sigma \dfrac{1}{n^s}$ for $\operatorname{Re}(s) > 1$. Prove the following:

(*i*) $\Gamma(s)\,\zeta(s) = \displaystyle\int_0^\infty G(t)\, t^s \frac{dt}{t}.$

(*ii*) $\zeta(s)$ is analytic except for a simple pole at $s = 1$.

(*iii*) The residue of $\zeta(s)$ at $s = 1$ is 1.

(*iv*) $\zeta(s)$ has zeros of order 1 at the negative integers.

11

EXTENSION OF THE MAXIMUM MODULUS PRINCIPLE

PART II

We first recall Weierstrass Factorization Theorem. The theorem states that every entire function f can be represented in the form

$$z^m e^{g(z)} \Pi \left(1-\frac{z}{z_n}\right) \exp \left(\frac{z}{z_n}+\frac{1}{2}\frac{z^2}{z_n^2}+\ldots+\frac{1}{k_n}\frac{z^{k_n}}{z_n^{k_n}}\right)$$

where $k_n \to \infty$ sufficiently fast and g is an entire function and the z_n are taken in order of increasing modulus. This representation is not very useful from the application point of view. We give below another representation which is more useful than Weierstrass factorisation theorem. The following Hadamard's representation is for entire functions that are of **finite order** in the following sense.

Let $M(r) = \max_{|z|=r} |f(z)|$; then f is said to be of order μ if μ is the greatest lower bound of numbers s for which

$$\lim\sup r^{-s} \log M(r) < \infty.$$

Roughly speaking, this means that $M(r)$ is not much larger than $\exp(r^{\mu})$, and sometimes about that large.

11.4 HADAMARD'S PRODUCT REPRESENTATION

If f is of order μ with zeros at 0 and z_n, then

$$f(z) = z^m e^{P(z)} \prod_{n=1}^{\infty} \left(1-\frac{z}{z_n}\right) \exp\left[\frac{z}{z_n}+\frac{1}{2}\left(\frac{z}{z_n}\right)^2+\ldots+\frac{1}{p}\left(\frac{z}{z_n}\right)^p\right]$$

where $p \leq \mu$ and $P(z)$ is a polynomial of degree at most p.

Examples (*i*): $\sin \pi z$ is of order 1; Hadamard's product with $p = 1$ is

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1-\frac{z^2}{n^2}\right).$$

This is called Euler's product for the sine function.

(*ii*) The reciprocal of the gamma function is also of order 1, its Hadamard product is

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{z_n}\right) e^{-z/n},$$

where γ is Euler's constant,

$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n\right)$$

(for the definition and other properties of the gamma function, see 11.8).

11.5 THE EFFECT OF ZEROS, JENSEN'S FORMULA

An entire function whose maximum modulus $M(r)$ increases rapidly as $r \to \infty$, may have very few zeros or even none at all. One can think of $\exp(z^n)$ or $\exp(e^z)$. On the other hand, if the function has too many zeros z_n in a disk of radius r, the product $\Pi\,(1 - z / z_n)$ may not converge. One can think of $z_n = n$. Hadamard's product representation quoted above suggests that if there are more zeros, the order of $M(r)$ will be larger. This may seem paradoxial because of our expectation that a function with many zeros may be small in modulus. The actual situation is quite different. There is a *principle* that says that when a function has many zeros, its modulus must frequently be large in order to compensate; if the modulus is not large, an analytic function with many zeros may vanish identically. We are familiar with one case (see Example page 81) when the principle can be quantified. In other words, if $M(r) = O(r^n)$ as $r \to \infty$, then the function is a polynomial of degree at most n and therefore at most n zeros. Based on the same principle we present below some more results.

Jensen's Theorem

Assume that f is analytic in a disk $|z| \leq R$, *but not identically zero. Assume also that* $f(0) \neq 0$. *Let f have zeros* $\{z_k\}$, $k = 1, 2, 3, \ldots, n$ *in* $|z| \leq R$, *and let* $f(z) \neq 0$ *on* $|z| = R$. *Then*

$$(1) \quad \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})|\, d\theta - \log |f(0)| = \sum_{j=1}^{n} \log \frac{R}{|z_j|}.$$

Proof: We have seen in Chapter 8 (see 8.3) that, if z_0 is a zero of f, then the function f'/f has a simple pole at z_0 with residue 1. It follows that

$$\frac{1}{2\pi i} \int_{|z|=R} g(z) \frac{f'(z)}{f(z)}\, dz = \Sigma\, g(z_n),$$

where g is analytic in $|z| \le R$ and z_n are the zeros of f inside $|z| = R$. This suggests that the right side of (1) is connected with

$$I = \frac{1}{2\pi i} \int_C (\log z) \frac{f'(z)}{f(z)}\, dz,$$

but $g(z) = \log z$ is not analytic. In order to remove this difficulty we consider the same integral when C is the circle $|z| = R$ with a cut along the positive real axis from 0 to R. If f has zeros on the cut, we replace $f(z)$ by $f(ze^{i\lambda})$ with λ chosen so that $f(ze^{i\lambda})$ has *no* zeros on the cut. Note that there exists such a λ, since there are only finitely many zeros to avoid; and the change does not affect the moduli of the zeros. The function $\log z$ is determined in the cut disk by taking $\log(-1) = \pi i$. It follows now that the integrand in I is analytic except for simple poles at the zeros of f inside C, and we obtain

$$I = \sum_{j=1}^{N} \log z_j.$$

On the other hand, we can write I as the sum of three integrals, one along the circle, one along the lower side of the cut, and one along the upper side. The integral along the upper side is

$$(2) \qquad \frac{1}{2\pi i} \int_0^R (\log x) \frac{f'(x)}{f(x)}\, dx;$$

the integral along the lower side is

$$(3) \qquad -\frac{1}{2\pi i} \int_0^R (\log x + 2\pi i) \frac{f'(x)}{f(x)}\, dx;$$

and their sum is

$$(4) \qquad -\int_0^R \frac{f'(x)}{f(x)}\, dx = \log f(0) - \log f(R).$$

The integral around the circle can be written in the following form:

$$(5) \qquad \frac{1}{2\pi i} \int_{|z|=R} [(\log z)]\, [\log f(z)]'\, dz.$$

Integrating (5) by parts, we have

$$(6) \qquad \frac{1}{2\pi i} (\log z) \log f(z) \Big|_{|z|=R} - \frac{1}{2\pi i} \int_{|z|=R} \log f(z) \frac{dz}{z}.$$

Putting the integrals (4) and (6) together and taking real parts, we have

$$(7) \qquad \sum_{j=1}^{N} \log |z_j| = \operatorname{Re} I = \log|f(0)| - \log|f(R)|$$
$$+ \operatorname{Re}\left\{\frac{1}{2\pi i} (\log z) \log f(z) \Big|_{|z|=R}\right\} - \frac{1}{2\pi} \int_0^{2\pi} \log f(\mathrm{R}e^{i\theta})\, d\theta.$$

We now need to compute change in $(\log z) \log f(z)$ around $|z| = R$.

When z goes around the circle $|z| = R$, $\log z$ starts as $\log R$ and ends as $\log R + 2\pi i$, whereas $\log f(z)$ starts as $\log f(R)$ and ends as $\log f(R) + 2\pi i N$, if there are N zeros. Thus the change in $[\log f(z)]$ $(\log z)$ around the circle $|z| = R$ is

$$(8) \qquad (\log R + 2\pi i)\,\{\log f(R) + 2\pi i N\} - (\log R)\,\{\log f(R)\}$$
$$= 2\pi i\,\{\log|f(R)| + N(\log R + 2\pi i)\}$$

Dividing (8) by $2\,\pi\,i$ and taking real parts, we have

$$(9) \qquad \operatorname{Re} \frac{1}{2\pi i}\,(\log z)\log f(z)\,\Big|_{|z|=R} = \log|f(R)| + N\log R$$

It now follows from (7) and (9) that

$$(10) \qquad \sum_{j=1}^{N} \log|z_j| = \log|f(0)| + N\log R - \frac{1}{2\pi}\int_0^{2\pi} \log|f(\mathrm{Re}^{i\theta})|\,d\theta,$$

that is,

$$(11) \qquad \frac{1}{2\pi}\int_0^{2\pi} \log|f(\mathrm{Re}^{i\theta})|\,d\theta - \log|f(0)| = \sum_{j=1}^{N} \log\frac{R}{|z_j|}.$$

This proves Jensen's theorem.

11.6 SOME CONSEQUENCES OF JENSEN'S THEOREM

Observe that if f has no zeros, then $\log f$ is harmonic and Jensen's theorem reduces to the mean value theorem for harmonic functions (see Theorem 4, Chapter 10). Jensen's formula also shows that if $|f(\mathrm{Re}^{i\theta})|$ is not large, there cannot be too many zeros in $|z| < R$.

Let $n(t)$ denotes the counting function of the zeros (that is, the number of zeros of modulus less than t). The right side of (1) (Jensen's formula) can be written as

$$\int_0^R t^{-1} n(t)\,dt.$$

This is due to the fact that $n(t)$ is a step function that jumps by k when t crosses a value for which the circumference $|z| = t$ contains k zeros. We consider the case when no circle $|z| = t$ contains more than one (simple) zero and the general case can be carried out in the same way. We have $n(0) = 0$, since $f(0) \neq 0$, and $n(t) = 0$ for $0 < t < |z_j|$. Then $n(t) = 1$ from $|z_1|$ to $|z_2|$, $n(t) = 2$ from $|z_2|$ to $|z_3|$, and so on. Thus

$$\int_0^R \frac{n(t)}{t}\,dt = \int_{|z_1|}^{|z_2|} \frac{dt}{t} + 2\int_{|z_2|}^{|z_3|} \frac{dt}{t} + \ldots + N\int_{|z_N|}^{R} \frac{dt}{t}$$
$$= \log|z_2| - \log|z_1| + 2(\log|z_3| - \log|z_2|)$$
$$+ \ldots + N(\log R - \log|z_N|)$$

$$= -\log|z_1| - \log|z_2| - \ldots - \log|z_N| + N\log R$$

$$= \log\frac{R}{|z_1|} + \log\frac{R}{|z_2|} + \ldots + \log\frac{R}{|z_N|}$$

Note that if $|f(0)| = 1$ and $|f(z)| \le M$ for $|z| \le R$, then

$$\int_0^R \frac{n(t)}{t}\,dt \le \max_{|z|=R} \log|f(z)|. \tag{12}$$

This inequality puts a restriction on the number of zeros that such a function can have in a disk $|z| < r < R$. This can be easily seen by considering the following simple case:

$$n(R/2)\log 2 = n(R/2)\int_{R/2}^{R}\frac{dt}{t}$$

$$\le \int_{R/2}^{R}\frac{n(t)}{t}\,dt \le \log M$$

In order to understand and give more precise statements regarding inequality (12) we solve below two exercises.

Exercise 1: Let f be an entire function and $|f(z)| \le Ae^{B|Z|}$ where A and B are constants, $B < 1$. Let f has zeros z_n with $|z_n| = n$ ($n = 1, 2, 3,\ldots$). Then $f(z) \equiv 0$.

Suppose that $f(z) \ne 0$. It follows from Jensen's formula that $f(0) \ne 0$,

$$\int_0^R \frac{n(t)}{t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta - \log|f(0)|,$$

where $n(t)$ is the number of zeros z_k with $|z_k| \le t$. If $f(0) = 0$, then we can consider the function $\frac{f(z)}{z^p}$, where p is the order of the zero at 0. Then Jensen's formula gives

$$\int_0^R \frac{n(t)}{t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta - p\log R - \log|f(0)|.$$

Since f has at least one zero of modulus k, $k = 1, 2, \ldots, [R]$, we may take roughly that $n(t)$ is at least t, so

$$\int_0^R t^{-1}\,n(t)\,dt \ge R,$$

and
$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta \le \log A + BR,$$

therefore we have a contradiction for large R if $B > 1$.

Using the same ideas as above we proceed as follows:

We have $n(t) = 0$ for $0 \le t < 1$, $n(t) \ge 1$ for $1 \le t < 2$, $n(t) \ge 2$ for $2 \le t < 3$, and so on. Hence $n(t) \ge [t]$ (the integral part of t), and

$$[R] - 1 - \log [R] = \int_1^{[R]} \frac{t-1}{t}\, dt \le \int_0^R \frac{n(t)}{t}\, dt$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} \log |f(Re^{i\theta})|\, d\theta - p \log R - \log \left| f(0) \right|.$$

Since $\log |f(Re^{i\theta})| \le \log A + BR$, it follows that

$$[R] - 1 - \log [R] \le \log A + BR - p \log R - \log f(0).$$

But this is impossible for large R if $B < 1$.

Exercise 2: Let f be an entire function with $\left| f(z) \right| \le A \exp(B\left|z\right|^2)$. Let f has at least n zeros on each circle $\left|z\right| = n$ $(n = 1, 2, 3, ...)$.

Then $f(z) \equiv 0$ provided $B < 1/4$.

Assume (as in the above Exercise 1) that $f(0) \ne 0$. Then

$$n(t) \ge 1 + 2 + 3 + ... + [t]$$

$$= \frac{[t]([t]+1)}{2} \text{ or about } \frac{t^2}{2}.$$

Hence $\int_0^R t^{-1}\, n(t)\, dt$ is approximately $\frac{1}{2}\int_0^R t\, dt = \frac{R^2}{4}$.

However,

$$\frac{1}{2\pi}\int_0^{2\pi} \log |f(\mathrm{R}e^{i\theta})|\, d\theta \le \log A + BR^2,$$

therefore we will get a contradiction if $B < 1/4$.

Using the same ideas as above we proceed as follows:

We have $$n(t) \ge \frac{(t-1)t}{2} \text{ for } R \ge t \ge 1,$$

$$\int_0^R \frac{n(t)}{t}\, dt \ge \frac{1}{2}\int_1^{[R]} (t-1)\, dt$$

$$\ge \frac{1}{4}([R]-1)^2,$$

whereas $$\frac{1}{4}([R]-1)^2 \le \frac{1}{2\pi}\int_0^{2\pi} \log |f(Re^{i\theta})|\, d\theta$$

$$\le \log A + BR^2.$$

We now consider the case when f has infinitely many zeros in $\left|z\right| < 1$. Suppose now that f is analytic and bounded in the disk $\left|z\right| < 1$. Let $\left|f(z)\right| \le M$, and number the zeros in order of increasing modulus. Then by Jensen's formula, when $R < 1$,

$$(13) \quad \sum_{\left|z_j\right| \le R} \log \frac{R}{\left|z_j\right|} \le |\log |f(0)|| + \frac{1}{2\pi}\int_0^{2\pi} \log |f(Re^{i\theta})|\, d\theta$$

$$\le |\log |f(0)|| + \log M.$$

Since the right side of (8) is independent of R, we can let $R \to 1$ and get the inequality

$$(14) \qquad \sum_{j=1}^{\infty} \log \frac{1}{|z_j|} \le |\log |f(0)|| + \log M.$$

Since $|z_j| \to 1$ as $j \to \infty$ (otherwise there would be a limit point of zeros inside the unit disk,) inequality (14) shows that the moduli of the zeros cannot approach 1 so slowly that the series would diverge. In other words, the series $\sum_{j=1}^{\infty} (1 - |z_j|)$ must converge. This can be seen easily since

$$\begin{aligned}\log \frac{1}{|z_j|} &= -\log [1 - (|z_j|)] \\ &= (1 - |z_j|) + \frac{1}{2}(1 - |z_j|)^2 + \cdots \\ &\ge 1 - |z_j|.\end{aligned}$$

11.7 PHRAGMEN-LINDELÖF THEOREM

In this section, we present some result which extend the Maximum Principle by easing the requirement of boundedness on the boundary.

We write a complex number in the form

$$\mathcal{S} = \alpha + it \text{ with real } \alpha, t.$$

Phragmen-Lindelöf Theorem

Let f be analytic in a strip $\alpha_1 \le \alpha \le \alpha_2$ *and bounded in absolute value by 1 on the sides of the strip. Let there be a number n* ≥ 1 *such that*

$$f(\mathcal{S}) = 0\left(e^{|s|^{\eta}}\right) \text{ in the strip.}$$

Then f is bounded by 1 in the whole strip.

Proof : By assumption, for sufficiently large $|t|$,

$$|f(\alpha + it)| \le e^{|t|^{\lambda}} \text{ where } \lambda > \eta.$$

Choose an integer $k \equiv 2 \pmod 4$ such that $k > \lambda$. If $S = re^{i\theta}$, then

$$\mathcal{S}^k = r^k (\cos k\theta + i \sin k\theta)$$

and $k\theta$ is near to π.

Consider the function

$$h_{\varepsilon}(\mathcal{S}) = h(\mathcal{S}) = f(\mathcal{S})\, e^{\varepsilon S^K}, \text{ with } \varepsilon > 0.$$

Then for S in the strip we have

$$\left|h(\mathcal{S})\right| \le e^{|t|^{\lambda}} e^{\varepsilon r^{k}} \cos k\theta$$

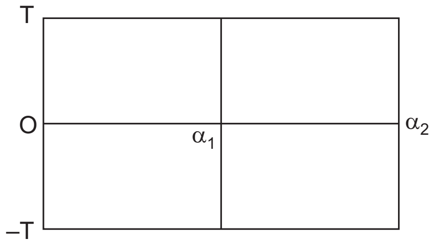


Fig. 11.I

It follows that for large T the function $h(\mathcal{S})$ is bounded by 1 on the horizontal segment $t = T$, $\alpha_1 \le \alpha \le \alpha_2$. It is also clear that $|h(\mathcal{S})|$ is bounded by 1 on the boundary of the rectangle, as shown in Fig. 11.I. Hence

$$\left|f(\mathcal{S})\right| \le e^{-\varepsilon r^{k}} \cos k\theta$$

inside the rectangle. This is true for $\varepsilon > 0$, and so

$$\left|f(\mathcal{S})\right| \le 1 \text{ inside the rectangle.}$$

This proves the theorem.

Our next theorem gives the batter result regarding the behaviour of the function, and so we assume that the function is analytic in a whole strip.

First Convexity Theorem

Let $\mathcal{S} = \alpha + it$. *Let f be analytic and bounded on the strip* $a \le \alpha \le b$ *for each* α. *Let*

$$M_f(\alpha) = M(\alpha) = \sup\left|f(\alpha + it)\right|.$$

Then log $M(\alpha)$ *is a convex function of* α.

Proof : We need to prove that

$$M(\alpha)^{b-a} \le M(a)^{b-\alpha} M(b)^{\alpha-a}.$$

Consider first the case when $M(a) = M(b) = 1$.

We shall prove that $M(\alpha) \le 1$.

Suppose that $\left|f(\mathcal{S})\right| \le K$ in the strip. For $\varepsilon > 0$, let

$$h_{\varepsilon}(\mathcal{S}) = \frac{1}{1 + \varepsilon(\mathcal{S} - a)}.$$

Then the real part of $1 + \varepsilon(\mathcal{S} - a)$ is ≥ 1. It follows that $|h_{\varepsilon}(\mathcal{S})| \le 1$. Also, for $t \ne 0$

$$\left|h_\varepsilon(S)\right| \le \frac{1}{\varepsilon\left|t\right|},$$

and therefore
$$\left|f(S)\, h_\varepsilon(S)\right| \le \frac{K}{\varepsilon\left|t\right|}.$$

Let ε be small, and choose $t = \pm\,\frac{K}{\varepsilon}$. On the boundary of the rectangle with sides at $\alpha = a$, $\alpha = b$ with top and bottom $\pm\,\frac{K}{\varepsilon}$, we find that $\left|fh_\varepsilon\right| \le 1$. Hence $\left|fh_\varepsilon\right| \le 1$ on the whole of the rectangle.

Letting $\varepsilon \to 0$ we get

$$\left|f\right| \le 1 \text{ on the strip.}$$

In order to consider the general case, let

$$g(S) = M(a)^{\frac{b-s}{b-a}}\; M(b)^{\frac{s-a}{b-a}}.$$

Then g is entire, has no zeros, and $1/g$ is bounded on the strip. It follows that

$$\left|g(a+it)\right| = M(a) \text{ and } \left|g(b+it)\right| = M(b)$$

for all t. Thus

$$M_{f|g}(a) = M_{f|g}(b) = 1.$$

Making use of the first part of the proof, it follows that

$$|f|\, g| \le 1.$$

Hence

$$\left|f\right| \le \left|g\right| \text{ and this completes the proof.}$$

Corollary (Hadamard Three Circle Theorem)

Let f be analytic on an annulus $\xi \le \left|z\right| \le \eta$, *centred at the origin. Let*

$$M(\rho) = \sup_{|z|=\rho} \left|f(z)\right|.$$

Then log $M(\rho)$ *is a convex function of log* ρ.

In other words,

$$\log\left(\frac{\eta}{\xi}\right) \log M(\rho) \le \log\left(\frac{\eta}{\rho}\right) \log(\xi) + \log\left(\frac{\rho}{\zeta}\right) \log M(\eta).$$

Proof: Let $\phi(s) = f(e^s)$. Then ϕ is analytic and bounded on the strip $a \le \alpha \le b$, where $e^a = \xi$ and $e^b = \eta$. Applying the first convexity theorem we get the result.

In order to state the next theorem we need to define the growth exponent of f. Let f be analytic in the neighbourhood of a vertical line $\alpha + it$, with fixed α, and assume that

$$f(\alpha + it) = 0\left(|t|^{v}\right)$$

for some positive number v. The infimum of all such v is called the growth exponent of f. Denote this by $\psi(\alpha)$. It follows that

$$f(\alpha + it) = 0\left(|t|^{\psi(\alpha)+\varepsilon}\right)$$

for every $\varepsilon > 0$, and $\psi(\alpha)$ is the least exponent which makes the inequality true.

Second Convexity Theorem

Let f be analytic in the strip $a \le \alpha \le b$. Assume that for each α, $f(\alpha + it)$ grows at most like a power of $|t|$, and let $\psi(\alpha)$ be the least number ≥ 0 for which

$$f(\alpha + it) = 0\left(|t|\right)^{\psi(\alpha)+\varepsilon}$$

for every $\varepsilon > 0$.

Assume also that

$$f(\alpha + it) = 0\left(e^{|t|^{\zeta}}\right)$$

in the strip, with ζ, $1 \le \zeta$. Then $\psi(\alpha)$ is a convex function of α. In particular, $\psi(\alpha)$ is continuous on $[a, b]$.

Proof : It follows from Phragmen-Lindelöf Theorem that there is a uniform K such that

$$f(\alpha + it) = 0\left(|t|^{K}\right) \text{ in the strip.}$$

Write

$$M_{\varepsilon}(s) = \frac{b-s}{b-a}\,[\psi(a) + \varepsilon] + \frac{s-a}{b-a}\,[\psi(b) + \varepsilon].$$

Note that $M_{\varepsilon}(s)$ is the formula for the straight line segment between $\psi(a) + \varepsilon$ and $\psi(b) + \varepsilon$. It can be easily seen that the function

$f(s)\,(-is)^{-M_{\varepsilon}(s)}$ is bounded on the strip. Since we get now that

$\psi(\alpha) \le M_{\varepsilon}(\alpha)$ for each α in the strip, and every $\varepsilon > 0$, the theorem follows. We conclude this section by observing that Phragmen-Lindelöf theorem gives bounds on the middle.

11.8 THE GAMMA FUNCTION

The gamma function is defined initially by

$$(15) \qquad \Gamma(z) = \int_0^{\infty} t^{z} e^{-t} \frac{dt}{t}, \ \operatorname{Re} z > 0.$$

In order to see that Γ is analytic for $Re\ z > 0$, we need to know that the integral converges uniformly on each compact subset of the right-hand half plane. The proof is not very interesting and so we omit them. We shall now establish the following useful result:

(16) $\Gamma(z+1) = z\Gamma(z)$ for $Re\, z > 0$.

In order to show that (16) holds, we consider

$$\int_0^R t^{z-1} e^{-t}\, dt.$$

Integrating by parts (differentiating e^{-t} and integrating t^{z-1}) we get

$$z^{-1} e^{-t} t^z \Big|_0^R + \frac{1}{z}\int_0^R t^z e^{-t}\, dt.$$

When $R \to \infty$, the integrated terms drop out because

$\lim\limits_{R\to\infty} e^{-R} R^z = 0$, and the integral becomes $\Gamma(z+1)$.

Thus we obtained

$$\Gamma(z+1) = z\Gamma(z), \text{ when } Re\, z > 0.$$

This is called the function equation for the gamma function. It is easy to check that when n is 0 or a positive integer, then $\Gamma(n+1) = n!$

Integrals like (15) are called Mellin transforms. We write $\frac{dt}{t}$ because this expression is invariant under "multiplicative translations". This phrase means: Let f be any function which is absolutely integrable on $0 < t < \infty$. Let c be a positive number. Then

$$\int_0^\infty f(ct)\frac{dt}{t} = \int_0^\infty f(t)\frac{dt}{t}.$$

This can be verified by the change of variables formula. Note that, by replacing t by nt where n is a positive integer,

(17) $$\frac{1}{n^s} = \int_0^\infty e^{-nt} t^s \frac{dt}{t}, \text{ for } Re\, s > 0.$$

$\sum\limits_{n=1}^{\infty} n^{-s}$ is called the Riemann zeta function of the complex variable s.

We shall now obtain a very useful formula for the gamma function. Let

(18) $$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\, dt, p > 0.$$

Setting $t = y^2$ in (18), we have

$$\Gamma(p) = 2\int_0^\infty y^{2p-1} e^{-y^2}\, dy.$$

Similarly, we have

$$\Gamma(p) = 2\int_0^\infty x^{2q-1} e^{-x^2}\, dx$$

Thus

(19) $$\Gamma(p)\,\Gamma(q) = 4\int_0^\infty\int_0^\infty x^{2q-1} y^{2p-1} e^{-(x^2+y^2)}\, dx\, dy$$

Now, writing the double integral in polar co-ordinates in (19), we get

$$\Gamma(p)\,\Gamma(q) = 4\int_0^\infty e^{-r^2}\, r\,dr \int_0^{\pi/2} (r\cos\theta)^{2q-1}(r\sin\theta)^{2p-1}\,d\theta$$

$$= 4\int_0^\infty r^{2p+2q-1}\, e^{-r^2}\,dr \int_0^{\pi/2} (\cos\theta)^{2q-1}(\sin\theta)^{2p-1}\,d\theta$$

$$= 2\Gamma(p+q)\int_0^{\pi/2} (\cos\theta)^{2q-1}(\sin\theta)^{2p-1}\,d\theta. \tag{20}$$

If we take $p = q = \frac{1}{2}$, we get

$$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}.$$

Next, let $x = \sin^2\theta$, $1 - x = \cos^2\theta$, $dx = 2\sin\theta\cos\theta\,d\theta$;

$$2\int_0^{\pi/2} (\cos\theta)^{2q-1}(\sin\theta)^{2p-1}\,d\theta = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx. \tag{21}$$

The right-hand integral in (21) is called the **beta function** $B(p, q)$.

Now let $x = \dfrac{y}{1+y}$, then

$$B(p, q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\,dy.$$

It now follows from (20) that

$$B(p, q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} \tag{22}$$

If $0 < p < 1$ and $q = 1 - p$, then

$$\int_0^\infty \frac{y^{p-1}}{1+y}\,dy = \Gamma(p)\,\Gamma(1-p). \tag{23}$$

The left-hand integral of (23) is equal to $\dfrac{\pi}{\sin \pi p}$ when $0 < p < 1$ (see worked out example 27). Hence we obtain

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z},\ 0 < z < 1. \tag{24}$$

In particular, we take $z = 1/2$ to get $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$. Since $\Gamma(1-z) = \infty$ when $z = 0, 1, 2, \ldots$, it follows that $\Gamma(z)$ is never 0.

Using (24) it can be seen that Γ is analytic on the line $Re\,z = 0$ except at $z = 0$. This is due to the fact that $(1 + z)$ is analytic and not zero for $Re\,z = 0$, and so is $\Gamma(-z) = -\pi / [\Gamma(1+z)\sin \pi z]$ except at $z = 0$.

(Stirling's Formula)

Stirling's formula is the asymptotic development of the gamma function. The following is the statement giving an exact error term:

$$(25)\quad \log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log(2\pi) - \int_0^{\infty} \frac{P_1(t)}{z+t}\, dt,$$

where $P_1(t) = [t] - t + \frac{1}{2}$, a periodic function of period 1 with average 0 over a period ($[t]$ denotes the largest integer $\leq t$).

We do not prove Stirling formula in this form. Stirling formula with expressions that involve factorials are proved at the end of the book (see Exercise 53). This formula is useful in probability and statistical mechanics.

We now prove another simple formula which is called Euler's summation formula.

Euler's Formula

Let f be any continuously differentiable function of a real variable.

Then

$$(26)\quad \sum_{k=0}^{n} f(k) = \int_0^n f(t)\, dt + \frac{1}{2}(f(n) + f(0)) + \int_0^n P_1(t) f'(t)\, dt.$$

Proof: Note that $P_1'(t) = 1$ for $t \neq$ an integer. Integrating by parts (with $u = P_1(t)$ and $dv = f'(t)$), we have

$$\int_{k-1}^{k} P_1(t) f'(t)\, dt = P_1(t) f(t) \Big|_{k-1}^{k} - \int_{k-1}^{k} f(t)\, dt$$

$$= \frac{1}{2}(f(k) + f(k-1)) - \int_{k-1}^{k} f(t)\, dt.$$

We take the sum from $k = 1$ to $k = n$. Adding the integral

$$\int_0^n f(t)\, dt \text{ and } \frac{1}{2}(f(n) + f(0))$$

we get the sum $\sum_{k=0}^{n} f(k)$ on the left side of (26).

We now apply Euler's formula for the function $f(t) = \log t$. We have

$$\log n! = \int_1^n \log t\, dt + \frac{1}{2} \log n + \int_1^n t^{-1} P(t)\, dt$$

Hence

$$n! \sim n^{n + (1/2)} e^{-n} \tau(n),$$

where $\lim\limits_{n \to \infty} \tau(n) = A$ exists, by using the reasoning used in proving the convergence of the integral of the type

$$\int_0^\infty \frac{\sin x}{x}\, dx.$$

Some more problems are solved on these topics and interested readers can see the solved exercises at the end of this book.

EXERCISES

1. Suppose that f is an entire function of order less than 2, that is,

$$|f(z)| \le A \exp(B|z|^{2-\varepsilon}), \ \varepsilon > 0,$$

where A and B are constants. Assume that $f(m+n) = 0$ for all integral m and n. Then prove that $f(z) = 0$.

2. Suppose $a_1, a_2, \ldots, a_n$ are the zeros and $b_1, b_2, \ldots, b_n$ are the poles of f in the disk $|z| \le R$. Prove that

$$\log R^{m-n} \left| \frac{b_1 \ldots b_n}{a_1 \ldots a_m} f(0) \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})|\, d\theta$$

3. Suppose that f is the entire function such that $|f(z)| \le c(\varepsilon) e^{\varepsilon|z|}$ for every $\varepsilon > 0$. Suppose also that f is bounded on the real axis. Then show that f is a constant.

4. Let V be the right-half plane. Let f be continuous on the closure of V and analytic on V. Suppose that there are constants $A > 0$ and $\alpha < 1$ such that

$$|f(z)| \le A\, e^{|z|^\alpha}$$

for all z in V. Suppose also that f is bounded by 1 on the imaginary axis. Then prove that f is bounded by 1 on V. Prove also that the result does not hold if $\alpha = 1$.

5. Find the radius of convergence of

$$\sum z^n\, \Gamma(n + 1/2) / n\,!$$

6. Prove that for $Re\, z \ge 0$,

$$\frac{\Gamma'(z)}{\Gamma(z)} = \int_0^\infty \left[\frac{e^{-t}}{t} - \frac{e^{-zt}}{1 - e^{-t}} \right] dt.$$

12

ELLIPTIC FUNCTIONS

We assume that the reader is familiar with the preliminary notions of **abstract algebra,** which are very essential for the general theory of **elliptic functions.** We begin with some basic definitions.

12.1 GROUPS

A **group** is a set G together with operation defined between $a, b \in G$ ((a o b) or ab) satisfying

(*i*) for all $a, b \in G$, $ab \in G$;

(*ii*) $a\,(bc) = (ab)\,c$;

(*iii*) there exists an element $e \in G$ (called the identity element) such that

$$a \cdot e = e \cdot a = a \text{ for all } a \in G;$$

(*iv*) to each element $a \in G$, there exists an element $a^{-1} \in G$ (called the inverse to a) such that $aa^{-1} = a^{-1}\,a = e$.

A set G satisfying just the first two **axioms** is called a **semi-group.** If G is a group and if for all $a, b \in G$,

$$ab = ba;$$

then G is called **abelian.**

A subset S of a group G is called a **subgroup** if S is itself a group with respect to the operation in G.

A **lattice** is the complex plane C is a subgroup which is free of dimension 2 over the set of integers, and which generates C over the reals.

Let w_1, w_2 be a basis for a lattice L over the set of integers. We write

$$L = [w_1, w_2].$$

Such a lattice is illustrated in Fig. 12.I

Let $L = [w_1, w_2]$ and let $\beta \in C$. The set consisting of all points

$$\beta + t_1 w_1 + t_2\, w_2,\ 0 \leq t_i \leq 1$$

is called a **fundamental parallelogram** with respect to the given basis for the lattice.

Note that we can also take the values $0 \le t_i < 1$ to define a fundamental parallelogram.

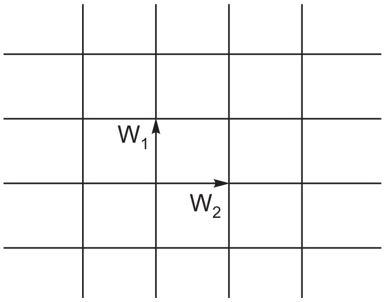


Fig. 12.I

If z_1 and z_2 are two complex numbers such that $z_2 - z_1 = m\,2w$, where $2w$ is fixed complex number and m is an integer. Then z_2 is called **congruent** to z_1 (**modulus** $2w$) and is denoted by

$$z_2 \equiv z_1 \ (\text{mod} \cdot 2w).$$

If z_1 and z_2 are two complex numbers such that $z_2 - z_1 = m\,2w_1 + n\,2w_2$, where w_1 and w_2 are complex numbers and m, n are integers, then z_2 is called congruent to z_1 (modulus $2w_1$, $2w_2$) and is denoted by

$$z_2 \equiv z_1 \ (\text{mod } 2w_1, 2w_2).$$

Let a function f be defined on a region Ω. If $f(z_1) = f(z_2)$ for all $z_1, z_2 \in \Omega$ where $z_2 - z_1 = m\,2w$ for at least one complex number $2w$ (m an integer), then the function f is called periodic with period $2w$.

Note that $\exp z$, $\exp mz$, $\sin z$ and $\sin \dfrac{2\pi}{l} z$ are periodic functions with period $2\pi i$, $\dfrac{2\pi i}{m}$, 2π and l respectively.

If the periods of a periodic function f are expressible in the form $m2w_1$, then f is called simply periodic.

If the periods of a periodic function f are expressible in the form $m2w_1 + n2w_2$, where m and n are integers, then f is called **doubly periodic.**

12.2 ELLIPTIC FUNCTIONS

An **elliptic function** is a meromorphic function on the complex plane C which is doubly periodic. In other words, an elliptic function f with respect to the lattice L is a meromorphic function on the complex plane C where

$$f(z + w) = f(z) \text{ for all } z \in C \text{ and } w \in L.$$

Observe that f is periodic iff

$$f(z + w_1) = f(z) = f(z + w_2).$$

We state some useful properties of elliptic function and the proof is left to the reader.

We assume that $0 < \arg (2w_2/2w_1) < \pi$.

Properties: (*i*) *The sum, difference, product and quotient of any two co-periodic elliptic functions are also elliptic with the same period.*

(*ii*) *The set of all co-periodic elliptic functions forms a field.*

(*iii*) *A rational function of co-periodic elliptic functions is an elliptic function with the same period.*

(*iv*) *The derivative of an elliptic function is an elliptic function with the same period.*

(*v*) *An elliptic function which is entire is a constant.*

Theorem 1. *Let P be a fundamental parallelogram for the lattice L. Let ∂P denotes the boundary of P. Suppose that the elliptic function f has no poles on ∂P. Then the sum of the residues of f in P is 0.*

Proof : Let P (Fig. 12.II) be a fundamental parallelogram.

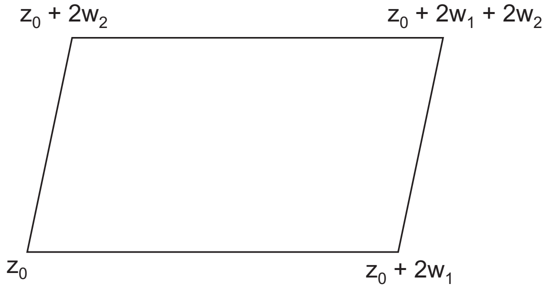


Fig. 12.II

By Cauchy's theorem, the sum of the residues

$$\Sigma \text{ Res } f = \frac{1}{2\pi i} \int_{\partial P} f(z)\, dz.$$

Now

$$\int_{\partial P} f(z)\, dz = \int_{z_0}^{z_0+2w_1} f(z)\, dz + \int_{z_0+2w_1}^{z_0+2w_1+2w_2} f(z)\, dz$$

$$+ \int_{z_0+2w_1+2w_2}^{z_0+2w_2} f(z)\, dz + \int_{z_0+2w_2}^{z_0} f(z)\, dz$$

$$= \int_{z_0}^{z_0+2w_1} f(z)\, dz + \int_{z_0}^{z_0+2w_2} f(z+2w_1)\, d(z+2w_1)$$

$$+ \int_{z_0+2w_1}^{z_0} f(z+2w_2)\, d(z+2w_2) + \int_{z_0+2w_2}^{z_0} f(z) dz$$

$$= \int_{z_0}^{z_0+2w_1} \{f(z) - f(z+2w_2)\}\, dz + \int_{z_0}^{z_0+2w_2} \{f(z+2w_1) - f(z)\}\, dz\,.$$

Since $f(z + 2w_1) = f(z), f(z + 2w_2) = f(z)$, it follows that

$$\int_{\partial P} f(z)\, dz = 0.$$

Thus the theorem is proved.

Theorem 2. *Let P* (Fig. 12.II) *be a fundamental parallelogram. Suppose that the elliptic function f has no zero or pole on ∂P. Let $\{z_i\}$ be the singular points of f inside P. Suppose also that the function f has order m_i at z_i. Then*

$$\Sigma m_i = 0.$$

Proof: Since f is an elliptic function, then f'/f is also an elliptic function. It follows from Theorem 1 that

$$0 = \frac{1}{2\pi i} \int_{\partial P} \frac{f'(z)}{f(z)}\, dz$$

$$= \text{sum of the residues of } \frac{f'(z)}{f(z)} \text{ at singularities inside } P.$$

$$= \Sigma m_i.$$

Theorem 3. *Let the hypotheses of Theorem 2 hold. Then $\Sigma\, m_i z_i \equiv 0$ (mod $\cdot$ $2w_1, 2w_2$).*

Proof: Since Res $z_i \dfrac{f'(z)}{f(z)} = m_i z_i$, then

$$\frac{1}{2\pi i} \int_{\partial P} \frac{z \cdot f'(z)}{f(z)}\, dz = \Sigma\, m_i z_i.$$

We now compute the integral over the boundary of the parallelogram.

$$\frac{1}{2\pi i} \int_{\partial P} z \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i} \left[\int_{z_0}^{z_0 + zw_1} z \frac{f'(z)}{f(z)}\, dz + \int_{z_0 + 2w_1}^{z_0 + 2w_1 + 2w_2} z \frac{f'(z)}{f(z)}\, dz \right.$$

$$\left. + \int_{z_0 + 2w_1 + 2w_2}^{z_0 + 2w_2} z \frac{f'(z)}{f(z)}\, dz + \int_{z_0 + 2w_2}^{z_0} z \frac{f'(z)}{f(z)}\, dz \right]$$

$$= \frac{1}{2\pi i} \left[\int_{z_0}^{z_0 + 2w_1} \{z - (z + 2w_2)\} \frac{f'(z)}{f(z)}\, dz + \int_{z_0}^{z_0 + 2w_2} \{z + 2w_1 - z\} \frac{f'(z)}{f(z)}\, dz \right]$$

$$= \frac{1}{2\pi i} \left[2w_1 \int_{z_0}^{z_0 + 2w_2} \frac{f'(z)}{f(z)}\, dz - 2w_2 \int_{z_0}^{z_0 + 2w_1} \frac{f'(z)}{f(z)}\, dz \right]$$

$$= \frac{1}{2\pi i} [2w_1 [\log f(z)]_{z_0}^{z_0 + 2w_2} - 2w_2\, [\log f(z)]_{z_0}^{z_0 + 2w_1}$$

$$= \frac{1}{2\pi i} (4\pi i m w_1 + 4\pi i\, n w_2)$$

$$= m 2w_1 + n 2w_2.$$

Hence, we conclude that

$$\Sigma\, m_i z_i \equiv 0 \ (\text{mod} \cdot 2w_1 \cdot 2w_2).$$

12.3 WEIERSTRASS' ELLIPTIC FUNCTIONS

In order to define Weierstrass' Elliptic function we need an important lemma.

Lemma: *If* $\mu > 2$, *then*

$$\sum_m \sum_n{}' \frac{1}{(2mw_1 + 2nw_2)^{\mu}}$$

converges absolutely. Note that the summation $\Sigma\,\Sigma'$ *extends over all positive and negative integers m, n not simultaneously equal to zero.*

Proof: Let S_k be the sum of the terms $\dfrac{1}{(2mw_1 + 2nw_2)^{\mu}}$. Note that we consider two alternatives *i.e.* either

$$m = \pm k, -k \le n \le k \text{ or } n = \pm k, -k \le m \le k.$$

$$S_k \le \sum_k \frac{1}{|(m2w_1 + n2w_2)|^{\mu}}$$

$$\le 8k \frac{1}{(kh)^{\mu}} \quad (h = \min(\,|2w_1|, |2w_2|)$$

$$= \frac{8}{h^{\mu}} \frac{1}{k^{\mu-1}}.$$

Hence, $\Sigma\, S_k$ is absolutely convergent for $\mu > 2$. Let

$$S_{mn} = \sum_{\substack{s \le m \\ t \le n}} \frac{1}{|s2w_1 + t2w_2|^{\mu}}.$$

Then $\quad S_{mn} \le \sum_1^N S'_k$

where S'_k is the sum obtained by adding absolute values of the terms of S_k and $k = \max(m, n)$. Thus the lemma is proved.

The **Weierstrass function** is defined by

$$\mathcal{P}(z) = \frac{1}{z^2} + \Sigma\Sigma' \left\{ \frac{1}{(z - \Omega_{mn})^2} - \frac{1}{\Omega_{mn}^2} \right\}$$

where $\Omega_{mn} \ne 0$, $\Omega_{mn} = m2w_1 + n2w_2 \equiv 0$ (mod, $2w_1$, $2w_2$). Note that the summation $\Sigma\Sigma'$ extends over all positive and negative integers m, n except $m = 0$, $n = 0$ simultaneously.

Observe that the series expression for $\mathcal{P}$ shows that it is meromorphic with pole of order two at Ω_{mn}.

We now show that the series for $\mathcal{P}(z)$ converges absolutely and uniformly for every z except $z = \Omega_{mn}$.

$$\left|\frac{1}{(z-\Omega_{mn})^2}-\frac{1}{\Omega_{mn}^2}\right| = \left|\frac{2\Omega_{mn}z-z^2}{\Omega_{mn}^2(z-\Omega_{mn})^2}\right|$$

$$\leq \frac{\{2\,|\,\Omega_{mn}\,|+|\,z\,|\}\,|\,z\,|}{|\,\Omega_{mn}\,|^4\,|\,1-(z\,/\,\Omega_{mn})\,|^2}$$

$$< \frac{3\,|\,\Omega_{mn}\,|\,|\,z\,|}{|\,\Omega_{mn}\,|^4\,|\,1-(z/\Omega_{mn})\,|^2}$$

provided $|\,z\,| \leq |\,\Omega_{mn}\,|$

$$\leq \frac{12}{|\,\Omega_{mn}\,|^3}\,|\,z\,|, \text{ provided } |\,z\,| \leq |\,\Omega_{mn}\,|\,.$$

Thus the series $\quad \Sigma\,\Sigma'\left\{\dfrac{1}{(z-\Omega_{mn})^2}-\dfrac{1}{\Omega_{mn}^2}\right\}$

is absolutely and uniformly convergent for $|\,z\,| \leq \dfrac{1}{2}\,|\,\Omega_{mn}\,|$.

Properties: (*i*) *The function* $\wp\,(z)$ *is an even function.*

Proof: $\wp(z) = \dfrac{1}{z^2} + \Sigma\Sigma''\left\{\dfrac{1}{(z-\Omega_{mn})^2}-\dfrac{1}{\Omega_{mn}^2}\right\} + \Sigma\Sigma''\left\{\dfrac{1}{(z-\Omega'_{mn})^2}-\dfrac{1}{\Omega'^2_{mn}}\right\}$

where $\Omega_{mn} = m2w_1 + n2w_2$, $\Omega'_{mn} = -\,mw_1 - n2w_2$; m is positive integer and n is any integer.

Note that the summation $\Sigma\Sigma''$ extends over all positive integers m, any integer n except $m = 0$, $n = 0$ simultaneously. Setting $-\,z$ for z in the above expression for $\wp(z)$, we find that $\wp(z) = \wp(-\,z)$.

(*ii*) *The function* $\wp\,(z)$ *is doubly periodic with periods* $2w_1$, $2w_2$.

Proof: $\quad \wp(z) = \dfrac{1}{z^2} + \Sigma\,\Sigma'\left\{\dfrac{1}{(z-\Omega_{mn})^2}-\dfrac{1}{\Omega_{mn}^2}\right\}$

$$= \frac{1}{z^2} + \frac{1}{(z-2w_1)^2} - \frac{1}{(2w_1)^2} + \Sigma\,\Sigma''\left\{\frac{1}{(z-\Omega_{mn})^2}-\frac{1}{\Omega_{mn}^2}\right\}.$$

Note that $\Sigma\Sigma''$ is a summation for all m, n except $m = 0$, $n = 0$ and $m = 1$, $n = 0$.

$$\wp(z+2w_1) = \frac{1}{z^2} + \frac{1}{(z+2w_1)^2} - \frac{1}{(2w_1)^2} + \Sigma\,\Sigma''\left\{\frac{1}{(z+2w_1-\Omega_{mn})^2}-\frac{1}{\Omega_{mn}^2}\right\}$$

$$= \frac{1}{z^2} + \Sigma\,\Sigma'\left\{\frac{1}{(z-\Omega_{mn})^2}-\frac{1}{\Omega_{mn}^2}\right\} = \wp\,(z).$$

Similarly, it can be proved that

$$\wp\,(z + 2w_2) = \wp\,(z).$$

Note that the function $\wp(z)$ is elliptic but a doubly periodic function need not be elliptic. For example, the function exp $[\wp(z)]$ is doubly periodic but not elliptic.

The derivative of $\wp$ is defined by

$$\begin{aligned}\wp'(z) &= -\frac{2}{z^3} - \Sigma\Sigma' \frac{2}{(z-\Omega_{mn})^3} \\ &= -2\ \Sigma\Sigma \frac{1}{(z-\Omega_{mn})^3}.\end{aligned}$$

Note that the summation $\Sigma\,\Sigma$ is extended over all positive and negative integral values of m and n and $m = 0$, $n = 0$.

Observe that $\wp'$ being the derivative of $\wp$ is an odd elliptic function with the same periods.

We now obtain the power series development of $\wp$ and $\wp'$ in the neighbourhood of $z = 0$ from which we shall get the algebraic relation holding between these two functions.

Theorem 4. *The function $\wp(z)$ is representable by the power series*

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} a_{2^k}\, z^{2^k}$$

where

$$a_{2^k} = (2k+1)\ \Sigma\Sigma'\, \Omega_{mn}^{-(2^k+2)}.$$

Proof: Note that the function $\wp(z) - \dfrac{1}{z^2}$ is analytic in the neighbourhood of $z = 0$ and so is expressible in a power series of z in $|z| = |2w|$ where $|2w| = \min(|2w_1|, |2w_2|)$.

Write

$$\begin{aligned}&\Sigma\Sigma'\left\{\frac{1}{(z-\Omega_{mn})^2} - \frac{1}{\Omega_{mn}^2}\right\} \\ &= \Sigma\Sigma'\left\{\frac{1}{\Omega_{mn}^2\left(1-\dfrac{z}{\Omega_{mn}}\right)^2} - \frac{1}{\Omega_{mn}^2}\right\} \\ &= \Sigma\Sigma'\left\{\frac{2z}{\Omega_{mn}^3} + \frac{3z^2}{\Omega_{mn}^4} + \ldots\right\} \\ &= \Sigma\Sigma'\left\{\sum_{v}(v+1)\frac{z^v}{\Omega_{mn}^{v+2}}\right\} \\ &= \Sigma\left\{(v+1)\,\Sigma\Sigma'\frac{1}{\Omega_{mn}^{v+2}}\right\} z^v \\ &= \sum_{v} a_v\, z^v\end{aligned}$$

where $$a_v = (v+1)\ \Sigma\Sigma'\,\frac{1}{\Omega_{mn}^{v+2}}.$$

But $\wp(z)$ is an even function, so $a_v = 0$ for all odd integral values of v. Hence, the result follows.

It follows from Theorem 4 that $\wp'(z)$ is represented by the power series

$$\wp'(z) = -\frac{2}{z^3} + 2a_2 z + 4a_4 z^3 + \ldots$$

where $$a_{2^k} = (2k+1)\ \Sigma\Sigma'\,\Omega_{mn}^{-(2^k+2)}.$$

Theorem 5. *Let* $$g_2 = 20a_2 = 60\ \Sigma\Sigma'\ \Omega_{mn}^{-4}$$

and $$g_3 = 28a_4 = 140\ \Sigma\Sigma'\ \Omega_{mn}^{-6}\ .$$

Then $$\wp'^2 = 4\wp^2 - g_2\wp - g_3.$$

Proof: We have $\wp(z) = \dfrac{1}{z^3} + a_2z^2 + a_4z^4$

and $$\wp'(z) = -\frac{2}{z^3} + 2a_2 z + 4a_4z^3 + \ldots$$

where $$a_2 = 3\ \Sigma\Sigma'\ \Omega_{mn}^{-4},\ a_4 = 5\ \Sigma\Sigma'\ \Omega_{mn}^{-6}, \text{ etc.}$$

Then $\wp'^2(z) = \dfrac{4}{z^6} - \dfrac{8a_2}{z^2} - 16a_4 +$ terms involving positive powers of z.

$\wp^3(z) = \dfrac{1}{z^6} + \dfrac{3a_2}{z^2} + 3a_4 +$ terms involving powers of z.

Now $20a_2\ \wp(z) = \dfrac{20a_2}{z^2} + 20a_2^2 z^2 + 20a_4\, a_2z^4 + \ldots + \ldots\ .$

Thus

(1) $\wp'^2(z) - 4\wp^3(z) + 20\, a_2\wp(z) = -\,28a_4 +$ terms containing powers of z.

The left side of (1) is an elliptic function of order zero and so is a constant. It now follows from equation (1) that

$C = -\,28a_4$ where C is a constant.

Hence $$\wp'^2(z) = 4\wp^3(z) - 20\, a_2\,\wp(z) - 28a_4$$

where $$a_2 = 3\ \Sigma\Sigma'\ \Omega_{mn}^{-4},\ a_4 = 5\ \Sigma\Sigma'\ \Omega_{mn}^{6}.$$

Writing $20\, a_2 = g_2$ and $28\, a_4 = g_3$, we obtain

(2) $\wp'^2 = 4\wp^3 - g_2\wp - g_3$.

12.4 THE ADDITION THEOREMS

Theorem 6. *The function* $\wp$ *satisfies the algebraic relation*

$$\begin{vmatrix} \wp'(u_1) & \wp(u_1) & 1 \\ \wp'(u_2) & \wp(u_2) & 1 \\ -\,\wp'(u_1+u_2) & \wp(u_1+u_2) & 1 \end{vmatrix} = 0$$

where u_1, u_2 *are complex numbers.*

Proof : Let u'_1, u'_2 be points in the parallelogram such that

$$u_1' \equiv u_1,\ u_2' \equiv u_2\ (\text{mod}\cdot 2w_1, 2w_2).$$

Define the function ψ by

$$\psi(z) = \wp'(z) - a\,\wp(z) - b.$$

Note that ψ is an elliptic function. Since $\wp$ and $\wp'$ have poles of order two and three respectively at the origin, so ψ has a pole of order three at the origin. It is clear that ψ must have three zeros in the fundamental parallelogram. Let the zeros be denoted by u_1', u_2', u_3' where

$$u_1' + u_2' + u_3' \equiv 0\ (\text{mod}\cdot 2w_1, 2w_2).$$

Thus, $\wp$ satisfies the equations

$$\wp'(u_1') - a\wp(u_1') - b = 0$$
$$\wp'(u_2') - a\wp(u_2') - b = 0$$
$$\wp'(u_3') - a\wp(u_3') - b = 0$$

Eliminating $-a$, $-b$, from the above equations, we get

$$\begin{vmatrix} \wp'(u_1') & \wp(u_1') & 1 \\ \wp'(u_2') & \wp(u_2') & 1 \\ \wp'(u_3') & \wp(u_3') & 1 \end{vmatrix} = 0.$$

Since $\quad u_1' \equiv u_1;\ u_2' \equiv u_2$; and $u_3' \equiv -(u_1' + u_2')\ (\text{mod}\cdot 2w_1, 2w_2)$,

the result follows.

Theorem 7. *The function $\wp$ satisfies the algebraic relation*

$$\wp(u_1 + u_2) = -\wp(u_1) - \wp(u_2) + \frac{1}{4}\left[\frac{\wp'(u_1) - \wp'(u_2)}{\wp(u_1) - \wp(u_2)}\right]^2$$

where u_1, u_2 are complex numbers.

Proof : Write

$$p_i = \wp(u_i),\ p_i' = \wp'(u_i)$$

where $\quad u_1 + u_2 + u_3 = 0.$

Proceeding similarly as in Theorem 6 we find that p_i, p_i' satisfy the equations

(3) $y - ax - b = 0$

and

(4) $y^2 = 4x^3 - g_2x - g_3$

where $\quad y = \wp'(u),\ x = \wp(u).$

Thus, p_i satisfy the equation

(5) $4x^3 - g_2x - g_3 - (ax + b)^2 = 0$

i.e. $\quad 4x^3 - a^2x^2 - (g_2 + 2ab)\,x - (g_3 + b^2) = 0.$

We have $\quad p_1 + p_2 + p_3 = \dfrac{a^2}{4},$

$$p_1p_2 + p_1p_3 + p_2p_3 = -\frac{g_2 + 2ab}{4},$$

and
$$p_1p_2p_3 = \frac{g_3 + b^2}{4}.$$

Solving (3), we obtain

$$a = \frac{p_1' - p_2'}{p_1 - p_2},\ b = \frac{p_1p_2' - p_2p_1'}{p_1 - p_2}.$$

Hence $p_1 + p_2 + p_3 = \dfrac{a^2}{4} = \dfrac{1}{4}\left(\dfrac{p_1' - p_2'}{p_1 - p_2}\right)^2.$

This proves the theorem.

Corollary: $\wp(2z) = -2\wp(z) + \dfrac{1}{4}\left[\dfrac{p''(z)}{p'(z)}\right]^2.$

12.5 THE WEIERSTRASS' ZETA FUNCTION

The **Zeta function** is defined by

$$(6)\quad \zeta(z) = \frac{1}{z} + \Sigma\Sigma'\left\{\frac{1}{(z - \Omega_{mn})} + \frac{1}{\Omega_{mn}} + \frac{z}{\Omega_{mn}^2}\right\}.$$

where $\Omega_{mn} = 2mw_1 + 2nw_2$ and m, n are integers, not simultaneously zero.

Note that the summation $\Sigma\Sigma'$ extends over all integers m, n except $m = 0$, $n = 0$ simultaneously.

It follows from the absolute and uniform convergence of the series for $\wp(z)$ that the series for $\zeta(z)$ in (6) converges absolutely and uniformly.

Observe that ζ is an odd function. Observe also that ζ and $\wp$ are connected by the relation

(7) $\wp(z) = -\zeta'(z)$.

Theorem 8. *The function ζ is represented by the power series*

$$\zeta(z) = \frac{1}{z} - \frac{a_2}{3}z^3 - \frac{a_4}{5}z^5 - \ldots - \frac{a_2^n}{(2n+1)}z^{2n+1} - \ldots$$

where $a_k = \Sigma\Sigma'(k+1)\,\Omega_{mn}^{-(k+2)}.$

Proof: $\zeta(z) = \dfrac{1}{2} + \Sigma\Sigma'\left\{\dfrac{1}{(z - \Omega_{mn})} + \dfrac{1}{\Omega_{mn}} + \dfrac{z}{\Omega_{mn}^2}\right\}.$

$$= \frac{1}{z} - \Sigma\Sigma'\left\{\frac{z^2}{\Omega_{mn}^3} + \frac{z^3}{\Omega_{mn}^4} + \ldots\right\}.$$

Since ζ is an odd function, hence, the coefficients of z^{2k} for $k = 1, 2, 3, \ldots$ in the above expression are zero. Thus the theorem is proved.

Properties: (*i*) $\zeta(kz, kw_1, kw_2) = k^{-1}\zeta(z, w_1, w_2)$, *where k is scalar.*

This follows from the definition of the function ζ.

(*ii*) $\zeta(z + 2w_1) = \zeta(z) + 2\eta_1$;

$\zeta(z + 2w_2) = \zeta(z) + 2\eta_2$;

where $\eta_1 = \zeta(w_1)$ and $\eta_2 = \zeta(w_2)$.

Proof : Observe that

$$\zeta'(z + 2w_1) - \zeta'(z) = -\wp(z + 2w_1) + \wp(z) = 0.$$

Integrating we obtain

$$\zeta(z + 2w_1) = \zeta(z) + 2\eta_1,$$

where
$$2\eta_1 = \zeta(-w_1 + 2w_1) - \zeta(-w_1) = 2\zeta(w_1)$$

Thus $\eta_1 = \zeta(w_1)$.

The second equality of (*ii*) follows similarly.

(*iii*) *Legendre Relation*

$$\eta_1 w_2 - \eta_2 w_1 = \frac{\pi}{2} i.$$

Proof : We integrate around a fundamental parallelogram *P* (Fig. 12.II). By Residue theorem,

$$\int_{\partial P} \zeta(z)\, dz = 2\pi i\ \Sigma \text{ residue of } \zeta.$$

Since ζ has residue 1 at 0 and no other pole in a fundamental parallelogram containing 0. Hence

$$\int_{\partial P} \zeta(z)\, dz = 2\pi i.$$

We now compute the integral $\int_{\partial P} \zeta(z)\, dz$.

$$\int_{\partial P} \zeta(z)\, dz = \int_{z_0}^{z_0 + 2w_1} \zeta(z)\, dz + \int_{z_0 + 2w_1}^{z_0 + 2w_1 + 2w_2} \zeta(z)\, dz + \int_{z_0 + 2w_1 + 2w_2}^{z_0 + 2w_2} \zeta(z)\, dz + \int_{z_0 + 2w_2}^{z_0} \zeta(z)\, dz$$

$$= \int_{z_0}^{z_0 + 2w_2} \{\zeta(z + 2w_1) - \zeta(z)\}\, dz - \int_{z_0}^{z_0 + 2w_1} \{\zeta(z + 2w_2) - \zeta(z)\}\, dz$$

$$= 4\eta_1 w_2 - 4\eta_2 w_1.$$

Hence $\eta_1 w_2 - \eta_2 w_1 = \frac{\pi}{2} i.$

Note that it follows from the property (*ii*) that ζ is not a doubly periodic, so it is not an elliptic function. It follows from **Legendre relation** that η_1, η_2 cannot be both zero simultaneously. The numbers η_1 and η_2 are called basic **quasi periods** of ζ.

12.6 THE WEIERSTRASS' SIGMA FUNCTION

The **sigma function** is defined by

$$(8) \qquad \sigma(z) = z \prod_{m,n} \left\{ \left(1 - \frac{z}{\Omega_{mn}}\right) \exp\left(\frac{z}{\Omega_{mn}} + \frac{1}{2}\frac{z^2}{\Omega_{mn}^2}\right) \right\}.$$

Note that multiplication is extended over all positive and negative integers m, n except $m = 0$, $n = 0$ simultaneously.

Properties: (*i*) $\sigma(kz, kw_1, kw_2) = k\sigma(z, w_1, w_2)$.

(*ii*) $\frac{d}{dz}[\log \sigma(z)] = \zeta(z)$.

(*iii*) *The function* σ *is represented by the power series*

$$\sigma(z) = z + C_1 z^5 + C_2 z^7 + \ldots + C_n z^{2n+3} + \ldots$$

where $$C_1 = -\frac{g_2}{240}, \; C_2 = -\frac{g_3}{840}, \ldots$$

The proof of (*i*) and (*ii*) follows from the definition of the function ζ.

Proof of (*iii*) : We have

$$\log \frac{\sigma(z)}{z} = \int_0^z \left\{\zeta(z) - \frac{1}{z}\right\} dz$$

$$= \int_0^z \left\{- a_2 \frac{z^3}{3} - a_4 \frac{z^5}{5} - \ldots a_2 n \frac{z^{2n+1}}{(2n+1)} - \ldots\right\} dz$$

$$= -\frac{a_2}{12} z^4 - \frac{a_4}{30} z^6 - \ldots$$

Thus $$\sigma(z) = z \exp[-z^4 Q(z)]$$

where $$Q(z) = \frac{a_2}{12} + \frac{a_4}{30} z^2 + \ldots$$

i.e., $$\sigma(z) = z \left\{1 - z^4 Q(z) + \frac{z^8}{2!} Q^2(z) + \ldots\right\}$$

$$= z - \frac{a_2}{12} z^5 - \frac{a_4}{30} z^7 - \ldots .$$

Hence $$\sigma(z) = z + c_1 z^5 + c_2 z^7 + \ldots + c_n z^{2n+3} + \ldots$$

where $$c_1 = -\frac{a_2}{12} = -\frac{g_2}{240}, \; c_2 = -\frac{a_4}{30} = -\frac{g_3}{840}.$$

(*iv*) σ *is an odd function*

This follows from the property (*iii*).

(*v*) *The function* σ *and* $\wp$ *are connected by the relation*

$$\wp(z) = -\frac{d^2}{dz^2}\{\log \sigma(z)\}.$$

This follows easily. For,

$$\wp(z) = -\frac{d}{dz}\,\zeta(z)$$

$$= -\frac{d^2}{dz^2}\,\{\log\sigma(z)\}.$$

Theorem 9. *The function* σ *satisfies the relation*

$$\sigma(z+\Omega_{mn}) = (-1)^{m+n+mn}\exp\left\{2\eta_{mn}\left(z+\frac{\Omega_{mn}}{2}\right)\sigma(z)\right\}$$

where $2\eta_{mn} = 2m\eta_1 + 2n\eta_2.$

Proof : It follows from property (*ii*) (12.5) that

$$\zeta(z+\Omega_{mn}) = \zeta(z) + 2\eta_{mn}$$

i.e.

$$\frac{\sigma'(z+\Omega_{mn})}{\sigma(z+\Omega_{mn})} = \frac{\sigma'(z)}{\sigma(z)} + 2\eta_{mn}.$$

Integrating, we get

$$\log\sigma(z+\Omega_{mn}) = \log\sigma(z) + 2\eta_{mn}z + C$$

where C is a constant.

Thus $$\sigma(z+\Omega_{mn}) = \exp\{(2\eta_{mn}z + C)\}\,\sigma(z)$$

$$= \exp\left\{2\eta_{mn}\left(z+\frac{\Omega_{mn}}{2}\right) + C'\right\}\sigma(z)$$

i.e.

$$\sigma(z+\Omega_{mn}) = A\exp\left\{2\eta_{mn}\left(z+\frac{\Omega_{mn}}{2}\right)\sigma(z)\right\}. \tag{9}$$

where $A = e^{C'}.$

We need to determine A.

If m, n are not even, then $\frac{\Omega_{mn}}{2}$ is not a period. Setting $z = -\frac{\Omega_{mn}}{2}$ in (9), we get

$$A = \frac{\sigma\left(\frac{\Omega_{mn}}{2}\right)}{\sigma\left(-\frac{\Omega_{mn}}{2}\right)} = -1.$$

If m, n are even, then $\frac{\Omega_{mn}}{2}$ is a zero of $\sigma(z)$. In this case, by using **L'Hospital's rule,** we get

$$A = \frac{\sigma'\left(\frac{\Omega_{mn}}{2}\right)}{\sigma'\left(-\frac{\Omega_{mn}}{2}\right)} = 1.$$

The function σ' is an even function and it has no zeros at $\frac{\Omega_{mn}}{2}$. Note that σ has simple zeros at Ω_{mn}. Hence,

$$\sigma(z+\Omega_{mn}) = (-1)^{m+n+mn} \exp\left\{2\eta_{mn}\left(z+\frac{\Omega_{mn}}{2}\right)\sigma(z)\right\}.$$

The following properties of the function σ are the special cases of Theorem 9.

Properties: (*vi*) $\sigma\,(z+2w_1) = -\exp\,\{2\eta_1\,(z+w_1)\}\,\sigma\,(z);$

$$\sigma\,(z+w_2) = -\exp\,\{2\eta_2\,(z+w_2)\}\,\sigma\,(z);$$

$$\sigma\,(z+2w_3) = -\exp\,\{2\eta_3\,(z+w_2)\}\,\sigma\,(z).$$

(*vii*) *If* $\psi\,(z) = \dfrac{\sigma(z-\beta)}{\sigma(z-\delta)}$

then $\psi\,(z+\Omega_{mn}) = \exp\,\{2\eta_{mn}\,(\delta-\beta)\}\,\psi\,(z).$

Theorem 10. *Let* β *be a complex constant other than the periods. Then*

$$\wp(z) - \wp(\beta) = \frac{\sigma(z+\beta)\,\sigma(z-\beta)}{\sigma^2(z)\,\sigma^2(\beta)}.$$

Proof: The function $\wp(z) - \wp(\beta)$ has zeros at β and $-\beta$, and has a double pole at 0. Hence

$$(10)\quad \wp(z) - \wp(\beta) = A\,\frac{\sigma(z+\beta)\,\sigma(z-\beta)}{\sigma^2(z)}$$

where A is a constant.

Multiplying (10) by z^2 and letting $z \to 0$, we find that

$$A = -\frac{1}{\sigma^2(\beta)}.$$

Note that $\dfrac{\sigma^2(z)}{z^2} \to 1$ and $z^2\,\wp(z) \to 1$ as $z \to 0$. Thus the theorem is proved.

EXERCISES

1. Prove that

$$\wp'\,(2z) = \frac{\wp''(z)}{\wp'(z)}\,\{\wp(z) - \wp(2z)\} - \wp'(z).$$

2. Prove that $\dfrac{\wp'(z+w_1)}{\wp'(z)} = -\left\{\dfrac{\wp(1/2w_1) - \wp(w_1)}{\wp(z) - \wp(w_1)}\right\}^2.$

3. Prove that

$$\wp(2z) - \wp(w_1) = \frac{\{\wp(z) - \wp(1/2w_1)\}^2\,\{\wp(z) - \wp(w_2 + 1/2w_1)\}^2}{\{\wp(z)\}^2}$$

4. Prove that if $3w$ is a period of $\wp(z)$ then

$$\{\wp(z)-\wp(w)\}\ \{\wp(z+w)+\wp(w)\}\ \{\wp(z+2w)-\wp(w)\}=\wp'^2(w).$$

5. Prove that

$$\wp(u+v)\,\wp(u-v)=\frac{\{\wp(u)\,\wp(v)+\frac{1}{4}g_2\}^2+\{\wp(u)+\wp(v)\}\,g_3}{\{\wp(u)+\wp(v)\}^2}$$

6. Prove the following

(*i*) $\wp(z)-\wp(2z)=\dfrac{1}{4}\dfrac{d^2}{dz^2}\{\log\wp'(z)\}.$

(*ii*) $\wp(2u)-\wp(2v)=-\dfrac{\wp'(u+v)\,\wp'(u-v)}{\{\wp(u+v)-\wp(u-v)\}^2}$

7. Prove that

$$2\zeta(2z)=\zeta(z)+\zeta(z-w_1)+\zeta(z+w_1+w_2)+\zeta(z-w_2).$$

8. Prove that $\dfrac{\sigma(nz)}{\{\sigma(z)\}^{n2}}$ is a doubly periodic. Also, prove that

$$\frac{\sigma(3z)}{\sigma^2(z)}=3\wp(z)\,\wp'^2(z)-\frac{1}{4}\wp'^2(z)$$

9. Prove the following

(*i*) $\dfrac{\wp''(z)}{\wp'(z)}=2\zeta(2z)-4\zeta(z)$

(*ii*) $3\zeta(3z)-9\zeta(z)$

$$=\frac{\wp'^3(z)}{\wp^4(z)-1/2\,g_2\,\wp^2(z)-g_3\,\wp(z)-1/48\,g_2^2}$$

10. Prove that

$$\frac{\wp'(z)-\wp'(\beta)}{\wp(z)-\wp(\beta)}=2\zeta(z+\beta)-2\zeta(z)\ -2\zeta(\beta).$$

Using this result prove that if $u_1+u_2+u_3=0$ then

$$\{\zeta(u_1)+\zeta(u_2)+\zeta(u_3)\}^2+\zeta'(u_1)+\zeta'(u_2)+\zeta'(u_3)=0$$

11. Prove that

$$\sigma(2z)=\frac{-2\sigma(z-w_1)\,\sigma(z-w_2)\,\sigma(z-w_3)\,\sigma(z)}{\sigma(w_1)\,\sigma(w_2)\,\sigma(w_3)}.$$

12. Prove that $\wp'(z)=\dfrac{\sigma(2z)}{\sigma^4(z)}.$

13. Prove that $\wp'(z)=\dfrac{2\sigma(z-w_1)\,\sigma(z-w_2)\,\sigma(z-w_3)}{\sigma^3(z)\,\sigma(w_1)\,\sigma(w_2)\,\sigma(w_3)}.$

14. Prove that $\dfrac{\sigma(2z)}{\sigma(z)}=2\sigma'^3-3\sigma(z)\,\sigma'(z)\,\sigma''(z)+\sigma^2(z)\,\sigma'''(z).$

15. The associated sigma functions $\sigma_j(z)$ are defined by

$$\sigma_j(z)=\exp(-\eta_r)\,z\,\frac{\sigma(z+w_j)}{\sigma(w_j)},\ (j=1,2,3).$$

Prove that the functions $\sigma_j(z)$ are even functions of z.

13

ANALYTIC CONTINUATION, DIFFERENTIAL EQUATIONS

13.1 ANALYTIC CONTINUATION

Let *f* be an analytic function defined in a region Ω_1 and let *g* be an analytic function defined in a region Ω_2. Suppose that Ω_2 includes Ω_1. If $g(z) = f(z)$ for all points of Ω_1, we say that *g* is the **analytic continuation** of *f* in the region Ω_2.

For example, the function $f(z) = \sum_{n=0}^{\infty} z^n$ is analytic at all points within the circle $|z| = 1$, the function $g(z) = \frac{1}{1-z}$ is analytic except at $z = 1$, and $g(z) = f(z)$ within $|z| = 1$. Thus *g* is the analytic continuation of *f* over the rest of the plane.

There are several methods of analytic continuation of which the simplest (but tedious) is by power series. We indicate below how this is done:

With every non-singular point of the original region there is associated a circle of convergence say, C_1 of the Taylor series about this point. This circle C_1 will pass through the nearest singularity of the function. By selecting a new point of expansion in C_1 another Taylor series with a new circle of convergence say, C_2 can be constructed. The process can of course be repeated indefinitely.

If the union of sets of all such circles does not extend beyond the boundary of the original region, this is called a **natural boundary** and analytic continuation across it is not possible. A simple example is furnished by

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + \ldots .$$

This series converges for $|z| < 1$ and this *f* satisfies the following functional equations

$$f(z^2) = f(z) - z$$
$$f(z^4) = f(z) - z - z^2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$f(z^{2^n}) = f(z) - \sum_{k=0}^{n-1} z^{2^k}$$

This shows that if $z = e^{i\theta}$ is a singularity of $f(z)$, the roots of $z^{2^n} = (e^{i\theta})^{2^n}$ are also singularities where n is any integer. Hence, the existence of a single singularity on the circle of convergence implies an infinite number of singularities in every θ interval. Thus $|z| = 1$ is a natural boundary of (1) and analytic continuation across $|z| = 1$ is therefore not possible.

If there is no natural boundary, functions which are defined and analytic in some region can frequently be extended to analytic functions in some larger region. In performing analytic continuations we must check whether the singularities are in the interior to any of the circles of convergence. The process of analytic continuation is unique and leads to the same value for a function at a point, since, the power series representation of the function about this point is unique. We shall discuss this in detail in 13.3 and 13.4.

13.2 CONTINUATION ALONG A PATH

A function element is an ordered pair (f, D), where D is an open disk and f is analytic on D. We say that (f_0, D_0) and (f_1, D_1) are direct continuations of each other if $D_0 \cap D_1 \neq \phi$, and $f_0(z) = f_1(z)$ for all $z \in D_0 \cap D_1$.

Let f_0 be analytic on a disk D_0 centred at z_0. Let γ be a continuous path whose initial point is z_0 and terminal point is z_n. Let γ be defined on the interval [0, 1] and let $0 = s_0 \leq s_1 \leq s_2 \leq \dots \leq s_n = 1$ be a partition of [0, 1]. Let D_i be a disk containing $\gamma(s_i)$ as shown in Fig.13.I.

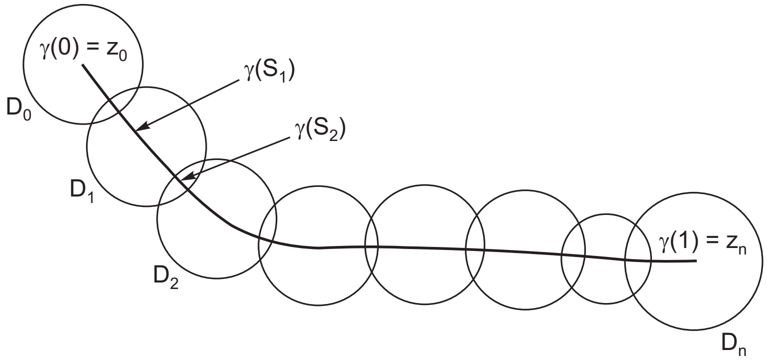


Fig. 13.I

Let $P = \{D_0, D_1, ..., D_n\}$ be finite sequence of disks such that $D_{i-1} \cap D_i \neq \phi$ for $i = 1, 2, ..., n$. We call P a chain. We say that the chain P cover the path γ if the image

$$\gamma([s_i, s_{i+1}]) \subset D_i, i = 0\ 1, 2, ..., n - 1.$$

If (f_0, D_0) can be continued along this chain P to (f_n, D_n), we say that (f_n, D_n) is an analytic continuation of (f_0, D_0) along the path γ.

Let γ be covered by the chain $P_1 = \{B_0, B_1, ..., B_m\}$ and the chain $P_2 = \{E_0, E_1, E_2, ..., E_n\}$ where $B_0 = E_0 = D_0$. Suppose that (f, D_0) is analytically continued along P_1 to an element (g_m, B_m) and along P_2 to an element (h_n, E_n). It can be easily verified that

$$g_m = h_n \text{ in } B_m \cap E_n.$$

Since B_m and E_n are disks with the same centre $\gamma(1)$, hence g_m and h_n have the same power series expansion about $\gamma(1)$. By replacing B_m and E_n by larger disk (whichever is the larger one of the two) we find that $g_m = h_n$.

Example: Let D_0 be a disk centred at $z_0 = 1$ and radius greater than zero but less than 1. Define $f_0(z) = \log z$ on D_0. Let γ be the circle of radius 1 oriented counterclockwise. Let (f_1, D_1) be the analytic continuation of $\log z$ along this γ. It can be easily seen that

$$f_1(z) = \log z + 2\pi i$$

near the point $z_0 = 1$. This shows that f_1 differs from f_0 by a constant, and is, not equal to f_0 near the point $z_0 = 1$.

13.3 CONTINUATION BY REFLECTION

Theorem 1 (Schwarz Reflection Principle). *Let f be analytic in a region Ω of the upper half z-plane. Let the boundary $\partial\Omega$ of Ω intersect the real axis in a single segment L. Let f be continuous on $\Omega \cup L$ and let f take only real values on L. If Ω^* is the reflection of Ω with respect to the real axis, then f can be continued analytically across L into Ω^*, by taking*

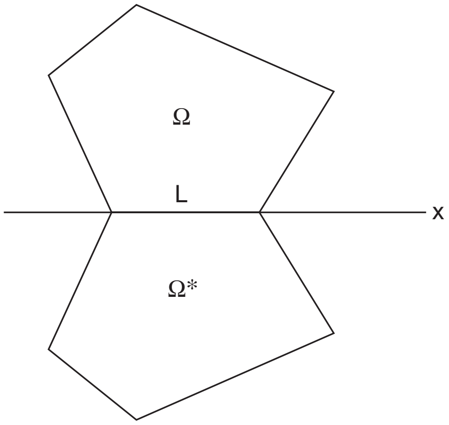


Fig. 13.II

$$f(z) = \overline{f(\bar{z})}\ (z\ \varepsilon\ \Omega^*)$$

(see Fig. 13.II).

We shall now prove a result that generalizes the Schwarz reflection principle.

Theorem 2. *With Ω and L as in the statement of Theorem 1, let f_1 be analytic in Ω and continuous in $\Omega \cup L$. Suppose that f_2 is analytic in Ω^* and continuous in $\Omega^* \cup L$. Suppose also that $f_1(z) = f_2(z)$ for all $z\ \varepsilon\ L$. Then f_2 is an analytic continuation of f_1 across L.*

Observe that Theorem 1 follows from Theorem 2 when $f_2(z) = \overline{f_1(\bar{z})}$.

Proof: In order to prove this theorem, we need to consider a disc D, centred on L, with its upper half in Ω, and prove that f can be continued analytically across the diameter L_1 of D. We will then have the situation (see Fig. 13.III), where L^* is a line segment parallel to L and so close to L that $\int_{L^*} f(z)\,dz$ and $\int_{L_1} f(z)\,dz$ are close to each other.

Note that this is possible since f is uniformly continuous in $\Omega \cup L$. If C is the upper half of the boundary ∂D of the disc D and C_1 is the part of ∂D above L^*, it follows from Cauchy's theorem that

$$\int_{L^* \cup C_1} f(z)\,dz = 0$$

Now we can make

$$\left| \int_{L^*} f(z)\,dz - \int_{L_1} f(z)\,dz \right| < \varepsilon$$

Hence,

$$\int_{L_1 \cup \partial D} f(z)\,dz = 0$$

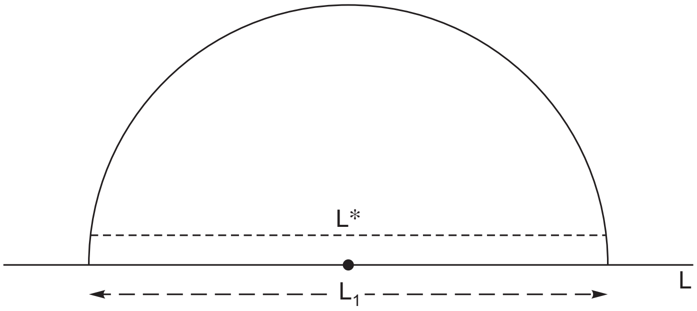


Fig. 13.III

Now let C^* be the lower half of ∂D and form

$$F(z) = \frac{1}{2\pi i} \int_{C \cup C^*} \frac{f(w)}{w - z}\,dw$$

(see Fig. 13.III).

Then F is analytic inside D. Suppose that $z \,\varepsilon\, D \cap \Omega$. Write

$$F(z) = \frac{1}{2\pi i}\left\{\left[\int_C \frac{f(w)\,dw}{w-z} + \int_{L_1} \frac{f(w)\,dw}{w-z}\right] + \left[\int_{C^*} \frac{f(w)\,dw}{w-z} - \int_{L_1} \frac{f(w)\,dw}{w-z}\right]\right\}$$

$$= f_1(z) + f_2(z).$$

Since $z \,\varepsilon\, D \cap \Omega$, we obtain

$$f_1(z) = f(z) \text{ and } f_2(z) = 0.$$

Similarly, when $z \,\varepsilon\, D \cap \Omega^*$, $f_1(z) = 0$, the reflection of $D \cap \Omega$ across L. It follows that F provides an analytic continuation of f from $D \cap \Omega$ to $D \cap \Omega^*$. The same argument applies to any other disc centred on L and hence f has an analytic continuation to Ω^*.

The following theorem tells us that we cannot make a direct analytic continuation under certain conditions.

Theorem 3 (Vivanti-Pringsheim). *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *has radius of convergence r and let* $a_k \geq 0$. *Then a direct analytic continuation of f from a point on (0, r) to any points of modulus greater than r is not possible.*

Note that when $a_k \geq 0$, *the point r is necessarily a singular point of f, even if* $\sum_{k=0}^{\infty} a_k r^k$ *converges. For a series* $\sum_{k=0}^{\infty} a_k (z-z_0)^k$, *the singular point will of course be at* $z_0 + r$. *(This point is not necessarily a singular point of the kind we have been discussing up to now.)*

Proof: We shall prove the result by contradiction.

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ has radius of convergence 1 and $a_k \geq 0$, and a direct analytic continuation past the point 1 is possible. Then the extended function would be analytic in some disc D centred at 1, and a disc centred at $\frac{1}{2}$, with radius r_1 just greater than $\frac{1}{2}$, would be contained in $D \cup \{|z| < 1\}$ (see Fig. 13.IV).

Then $$f^{(n)}\left(\frac{1}{2}\right) = \sum_{k=n}^{\infty} \frac{k!}{(k-n)!}\, a_k \left(\frac{1}{2}\right)^{k-n},$$

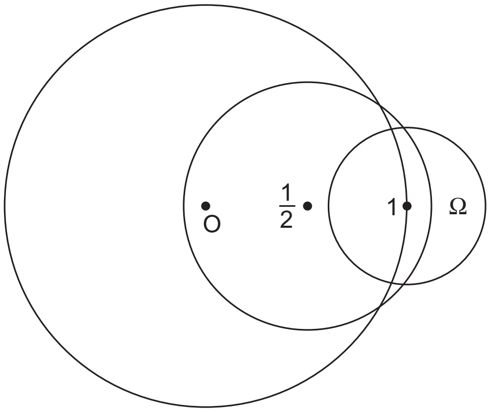


Fig. 13.IV

and the Taylor series of f about $\frac{1}{2}$ for a real $x > 1$ would be

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1/2)}{n!}\left(x - \frac{1}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(x - \frac{1}{2}\right)^n \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} a_k \left(\frac{1}{2}\right)^{k-n}$$

Since everything is positive in the above expression, it follows that

$$(*) \qquad f(x) = \sum_{k=0}^{\infty} a_k \sum_{n=0}^{k} \frac{1}{n!}\left(x - \frac{1}{2}\right)^n \frac{k!}{(k-n)!}\left(\frac{1}{2}\right)^{k-n}.$$

Note that the inner series in above equation (*) is the binomial series for

$$\left[\left(x - \frac{1}{2}\right) + \frac{1}{2}\right]^k = x^k,$$

therefore $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $k > 1$.

But this is the original power series for f, now with $x > 1$, although we assumed to begin with that f was not analytic in any disc of radius greater than 1, centred at 0. Thus we got a contradiction and this shows that f cannot be continued directly across $x = 1$.

Observe that the argument is unaffected if we assume that all the coefficients, except for a finite number, are positive.

The proof of the Vivanti-Pringsheim theorem is easy if we assume that $\sum_{k=0}^{\infty} a_k$ diverges. The following example shows this.

Example: Suppose that $\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence 1. If $a_k \geq 0$ and if $\sum_{k=0}^{\infty} a_k$ diverges, then $f(x) \to \infty$ as $x \to 1$ along the radius (0, 1).

Proof: Since $a_k \geq 0$ and $\sum_{k=0}^{\infty} a_k$ diverges, it follows that

$$\sum_{k=0}^{N} a_k > M$$

where M is a given large number and N is large enough.

Now, if $0 < x' < 1$,

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \geq \sum_{k=0}^{N} a_k x^k \geq x^N \sum_{k=0}^{N} a_k$$
$$> Mx^N$$

By taking x sufficiently close to 1, we have

$$f(x) > \frac{M}{2}.$$

But M is arbitrary, so $f(x) \to \infty$.

We now give an illustration which can be expressed by saying that f must have at least one singular point on its circle of convergence.

Example: (*i*) If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in $|z| < R$ and also at $z = r$. Then it is always possible to make a direct analytic extension of f to some points outside $|z| \leq r$.

(*ii*) Continuing as above in (*i*), prove that if f is analytic in $|z| \leq r$, this cannot be the disc of convergence of f.

Proof: (*i*) If f is analytic at $z = r$, then f is analytic in some neighbourhood of the point r. If we expand f about any point x_0 with $0 < x_0 < r$, then a disc $|z - x_0| < r - x_0 + \delta$ will be in the region of analyticity of f where $\delta > 0$, is sufficiently small (see Fig. 13.V).

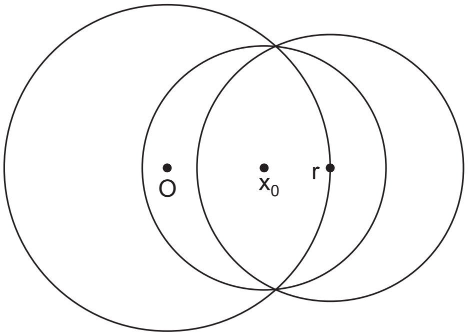


Fig. 13.V

(*ii*) If f is analytic in $|z| \leq r$, it is analytic in an open disc about each point of the circumference given by $|z| = r$. It follows from the Heine-Borel theorem that a finite number of the discs cover the circumference $|z| = r$; their union is an open set containing the trace of $|z| = r$ and therefore contains a circle, centred at 0, of radius greater than r.

13.4 NOWHERE-CONTINUABLE POWER SERIES

We have just seen that the sum of power series $\sum_{k=0}^{\infty} a_k z^k$ with $a_k \geq 0$ cannot be continued directly past the real positive point on the circle of convergence. Now we shall be concerned with conditions that make it impossible to continue the sum of a power series beyond the disc of convergence in any direction at all. There are functions that are analytic for $|z| < r$ but cannot be continued analytically outside this disc. Such a function is said to have a **natural boundary** on the circle $|z| = r$. We shall first verify that such functions do exist.

Example: Prove that $f(z) = \sum_{n=1}^{\infty} z^{n!}$ cannot be continued outside the unit disc.

Note that the Vivanti-Pringsheim theorem shows that f cannot be continued past the point 1. Now consider $f(ze^{in!\,\pi p/q})$, where p and q are integers. When n is sufficiently large, $n!\,\pi\,p/q$ is an even integer and

$$e^{in!\,\pi p/q} = 1$$

Hence, the power series represented by $f(re^{in!\,\pi p/q})$, has positive coefficients when n is sufficiently large. Then $f(ze^{in!\,\pi p/q})$, cannot be continued past 1 on the radius $z = x$. In other words, f cannot be continued along the ray $\arg z = \pi\,p/q$. Since these rays intersect the unit circumference in a dense set, f cannot be analytic at any point of the circumference.

We shall now introduce some new concepts which will be useful in understanding Hadamard's gap theorem.

If a function is defined by a power series with a finite radius of convergence, then we can extend the function beyond the disc of convergence by grouping the terms of the series. This can be done by considering only a subsequence of the partial sums. A power series for which this can be done is said to be **overconvergent**. We shall now construct an example of an overconvergent power series and the key to the construction is the following lemma:

Lemma: *Let* $\{n_k\}$ *be a sequence of positive integers such that* $n_{k+1}/n_k \geq \lambda > 1$ *and let p be a positive integer such that* $p > \dfrac{1}{\lambda - 1}$. *Then the polynomial* $p_k(z) = [z^p(1+z)]^{n_k}$, *when expanded in powers of z, contains no power of z that appear in any other* P_j $(j \neq k)$.

Proof : Note that the highest power of z in $P_k(z)$ has exponent $pn_k + n_k$; the lowest power in $P_{k+1}(z)$ has exponent pn_{k+1}. Since $n_{k+1}/n_k > \lambda > (1/p) + 1 = \dfrac{p+1}{p}$, it follows that the difference of these exponents is $pn_{k+1} - (p+1)n_k > 0$.

In order to construct an example of an overconvergent power series we define a function f by

$$f(z) = \sum_{k=1}^{\infty} a_k P_k(z),$$

where $\dfrac{1}{a_k}$ is the coefficient with largest modulus in the binomial expansion of $(1-z)^{n_k}$. It follows from Lemma (with $-z$ instead of z) that $P_k(z)$ contains no powers that appear in any other $P_j(z)$. Thus, if we replace each $P_k(z)$ by its expansion in powers of z, we get a power series $\Sigma c_k z^k$ whose partial sums of order $(p+1)\, n_k$ are just the partial sums of the series $\Sigma P_k(z)$. Observe that $\Sigma\, c_k z^k$ has an infinite number of terms with coefficient 1, and none with coefficient of modulus greater than 1. It follows that the radius of convergence of $\Sigma\, c_k z^k$ is 1. We now show that $\Sigma P_k(z)$ converges in a set that extends outside the circle $|z| = 1$; this will mean that a sequence of partial sums of $\Sigma\, c_k z^k$ (namely, those of index n_k) converges in a set that contains points z with $|z| > 1$; in other words, this will mean that $\Sigma\, c_k z^k$ is over convergent. This will be possible if there are points ξ outside the unit disc for which $|\xi^p(1-\xi) < \alpha < 1$, since then $\Sigma\, a_k P_k(\xi)$ will be dominated by $\Sigma\, |a_k|\, \alpha^{n_k}$ with $|a_k| \leq 1$. The existence of such points is obvious, since if

$|1 - \xi| < \varepsilon\ (0 < \varepsilon < 1)$, then

$|\xi^p(1-\xi)| < |\xi|^P \varepsilon < a$ provided $|\xi| < (\alpha/\varepsilon)^{1/p}$.

This allows $|\xi| = 1 + \delta$ with $\delta > 0$ if $1 > \alpha > \varepsilon$.

Hence, we conclude that the series $\Sigma\, c_k z^k$ is overconvergent in a neighbourhood of the point 1 provided that $\dfrac{n_{k+1}}{n_k} > \lambda > 1$ and $p > \dfrac{1}{\lambda - 1}$. The series $\Sigma\, c_k z^k$ has long gaps. The ratio of the lowest exponent following a gap to the highest exponent preceding that gap is

$$\left(\frac{n_{k+1}}{n_k}\right)\left(\frac{p}{p+1}\right) > 1.$$

Note that the sequence of partial sums that converges outside the unit disc consists of the partial sums that end at a gap. Note also that the existence of long gaps is necessary for overconvergence, in the sense that an overconvergent series (whose radius of convergence is 1) is always the sum of a series that converges in a larger disc and a series with long gaps.

We illustrate the above mentioned fact by an example.

Example: If $p = 1$ and $\lambda > 2$, then the overconvergent series just constructed has a sequence of partial sums that converges at least in

$$S = |z| < 1 \cup |z - 1| < 1$$

(see Fig. 13.VI).

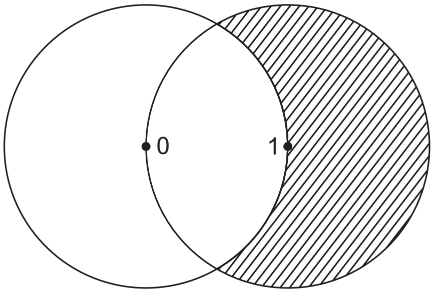


Fig. 13.VI

Since $p = 1$ and $\lambda > 2$, it follows that $p > \dfrac{1}{\lambda - 1}$. We need to show that

$$\Sigma\, a_k P_k(z) = \Sigma\, a_k [z(1-z)]^{n_k}$$

converges when $|1 - z| < 1$.

We know that the series converges for $|z| < 1$; the series also converges for $|1 - z| < 1$, since the series is unchanged if z is replaced by $1 - z$.

We now state and prove Hadamard's gap theorem.

Theorem 4 (Hadamard's Gap Theorem). *If* $a_n = 0$ *except for* $n = n_k$, *where* $\dfrac{n_{k+1}}{n_k} \geq \lambda > 1$, *then* $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, *with a finite radius of convergence, cannot be continued beyond the circle of convergence.*

Proof: Suppose that the radius of convergence of $\sum\limits_{n=0}^{\infty} a_n z^n$ is 1. Since $f(ze^{i\alpha})$ has the same gaps in its power series as f does, it is sufficient to prove that f cannot be analytic at $z = 1$.

Assume that f is analytic at $z = 1$, we need to arrive at a contradiction. Consider now the function

$$w(z) = f\left\{\left[\frac{1}{2} z^p (1+z)\right]^{n_k}\right\}$$

with $p > \lambda/\lambda + 1$.

It follows from Lemma that each $\left|\dfrac{1}{2} z^p\,(1+z)\right|^{n_k}$ expands into a sum of powers of z that do not occur in the expansions of the corresponding sum for any other value of n_k. Hence, the series by which we defined w is just the Maclaurin series of w with its terms grouped. It follows that w is analytic whenever Maclaurin series of w converges.

Now, since $\Sigma a_k z^k$ has radius of convergence 1, the set E where w is analytic contains at least the set where

$$\left| \frac{1}{2} z^p (1+z) \right| < 1$$

This set contains at least the intersection of the sets where $|z| < 1$ and $|z+1| < 2$. Note that the second of these sets contains all the points of the disc $|z| \leq 1$, except for $z = 1$. But by our assumption f is analytic at $z = 1$, the point 1 also belongs to E, since $\frac{1}{2} z^p (z+1) = 1$ at $z = 1$. Since E contains a disc of positive radius ε centred at 1, and hence a disc, centred at 0, of radius $1 + \delta > 1$ (see Fig. 13.VII). It follows that the Maclaurin series of w has radius of convergence greater than 1 and hence that

$$\Sigma a_k \left[\frac{1}{2} r^p (r+1) \right]^{n_k}$$

converges for some $r > 1$. Since $\frac{1}{2} r^p (r+1) > 1$ when $r > 1$, it follows that $\Sigma a_k \zeta^k$ converges for some $\zeta > 1$. This is a contradiction since this series had the radius of convergence 1. Hence the theorem is proved.

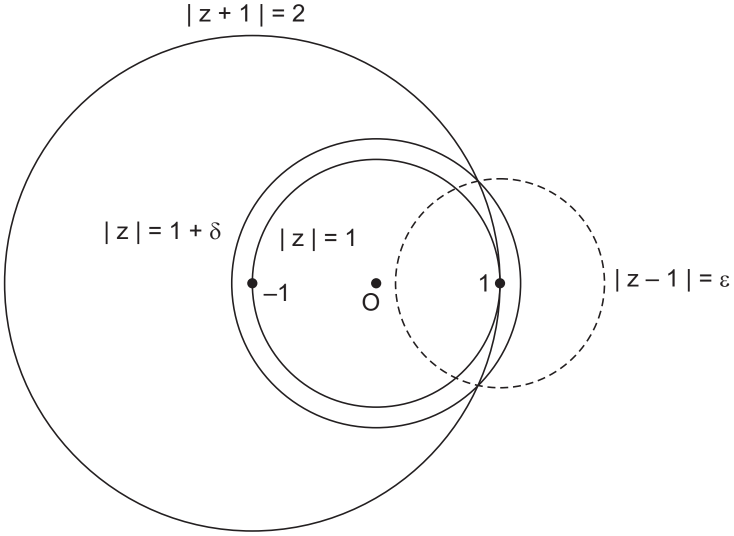


Fig. 13.VII

13.5 DIFFERENTIAL EQUATIONS

In the preceding discussion we found means to extend the domain of definition of an analytic function. In fact, one can define analytic function in a broader set up and we refer (for detailed discussion) the last chapter "Global Analytic Functions" of Ahlfors [1].

The theory of global analytic functions makes it possible to study, with a great degree of generality, the complex solutions of ordinary linear differential equations. Keeping in view the application of the topic we confine ourselves to ordinary linear homogeneous differential equations of the second order. These are of the form

$$(2) \qquad w''(z) + a(z)\, w'(z) + b(z)\, w(z) = 0.$$

If $(z - z_0)\, a(z) = p(z)$ and $(z - z_0)^2\, b(z) = q(z)$ are analytic at $z = z_0$, this point is said to be a regular singularity of (2) if

$$(3) \qquad |\, p(z_0)\,| + |\, q(z_0)\,| + |\, q'(z_0)\,| > 0.$$

The point $z = z_0$ is said to be an ordinary point of (2) if

$$(4) \qquad |\, p(z_0)\,| + |\, q(z_0)\,| + |\, q'(z_0)\,| = 0.$$

If $p(z)$ or $q(z)$ have a singularity at $z = z_0$, the point z_0 is said to be an **irregular singularity** of (2).

By using the methods of series substitutions for $w(z)$ we will solve (2).

There is no loss of generality if we take $z_0 = 0$.

Write

$$(5) \qquad w(z) = z^{\alpha} \sum_{k=0}^{\infty} w_k\, z^k$$

where

$$(6) \qquad p(z) = za(z) = \sum_{k=0}^{\infty} p_k\, z^k;$$

$$(7) \qquad q(z) = z^2 b(z) = \sum_{k=0}^{\infty} q_k\, z^k.$$

Multiplying (2) by z^2 and substituting $\zeta = \ln z$, (2) becomes

$$(8) \qquad w''(e^{\zeta}) - w'(e^{\zeta}) + p(e^{\zeta})\, w'(e^{\zeta}) + q(e^{\zeta})\, w(e^{\zeta}) = 0$$

and (5) – (7) become

$$(9) \qquad w(e^{\zeta}) = \sum_{k=0}^{\infty} w_k\, e^{(k+\alpha)\zeta}$$

$$(10) \qquad p(e^{\zeta}) = \sum_{k=0}^{\infty} p_k e^{k\zeta}$$

$$(11) \qquad q(e^{\zeta}) = \sum_{k=0}^{\infty} q_k\, e^{k\zeta}$$

and

$$(12)\qquad w'(e^{\zeta}) = \sum_{k=0}^{\infty} (k+\alpha)\, w_k\, e^{(k+\alpha)\,\zeta}$$

$$(13)\qquad w''(e^{\zeta}) = \sum_{k=0}^{\infty} (k+\alpha)^2\, w_k\, e^{(k+\alpha)\zeta}.$$

Taking the Cauchy product of (10) with (12) and (9) with (11) we get

$$\sum_{k=0}^{\infty} \left\{(k+\alpha)^2 w_k - (k+\alpha) w_k + \sum_{m=0}^{k} [(m+\alpha) p_{k-m}\, w_m + q_{k-m}\, w_m]\right\} e^{(k+\alpha)\zeta} = 0.$$

Hence, for $k \geq 1$

$$(14)\qquad [(k+\alpha)^2 + (p_0 - 1)(k+\alpha) + q_0]\, w_k = -\sum_{m=0}^{k-1} [(m+\alpha) p_{k-m} + q_{k-m}] w_m$$

and for $k = 0$

$$[\alpha^2 + (p_0 - 1)\,\alpha + q_0]\, w_0 = 0.$$

Since (2) is second order differential equation, there should be two arbitrary constants in the general solution of equation (2). Suppose that w_0 is one of them.

Then

$$(15)\qquad \alpha^2 + (p_0 - 1)\,\alpha + q_0 = 0.$$

This is called the **indicial equation.** The two values of α say, α_1 and α_2 determined by (15) are called the exponents of the regular singularity. Write

$$\varphi(\alpha) = \alpha^2 + (p_0 - 1)\alpha + q_0.$$

Then (14) becomes

$$(16)\qquad \varphi(k+\alpha)\, w_k = -\sum_{m=0}^{k-1} [(m+\alpha) p_{k-m} + q_{k-m}] w_m.$$

It follows that any w_k can be determined in terms of $\{p_k\}$, $\{q_k\}$ and the preceding w_k provided

$$(17)\qquad \varphi(k+\alpha) \neq 0 \text{ for } k > 0.$$

If the indicial equation does not have a double root ($\alpha_1 \neq \alpha_2$) and (17) holds, (16) will give a distinct solution for each root of (15).

If $w_0 = 0$ where w_0 is not arbitrary, (14) gives

$$[(1+\alpha)^2 + (p_0 - 1)(1+\alpha) + q_0] w_1 = 0$$

and for w_1 arbitrary, the same indicial equation (15) is satisfied by $(1+\alpha)$. Continuing in this way, it can be easily seen that the indicial equation must hold if there are any arbitrary constants in the series solution (5). It follows that there will be one arbitrary constant in each distinct solution. Thus there will be two arbitrary constants in the general solution of (2).

When $4\, q_0 = (p_0 - 1)^2$, the roots of the indicial equation (15) become

$$(18) \qquad \alpha = \frac{1 - p_0}{2}.$$

It now follows that for this α, (16) can be used to successively determine the coefficients in (5) for one solution with one arbitrary constant.

Again if the roots of the indicial equation (15) differ by an integer, (16) can be used to determine successively the coefficients in (5) for one solution. Note that α is equal to the root with the larger real part.

In each of these cases (7) cannot be used to find a second solution of (2) where α is given by (18). Observe that in the case where the roots differ by an integer, the successive determinations is not possible when $\varphi(k + \alpha) = 0$. In both these cases a first solution can be found. Denote this first solution by $w_{f_1}(z)$. By the help of the first solution, the second solution can also be determined. We indicate below how this is done.

Take $\qquad w(z) = f(z)\, w_{f_1}(z)$ in (2).

A little calculation will show that

$$r(z) = A + B \int \exp\left\{-\int a dz\right\} \frac{dz}{w_{f_1}^2(z)}$$

where A and B are constants of integration. $A = 1$, $B = 0$ gives the original solution, while $A = 0$, $B = 1$ gives a new second solution, say $w_{f_2}(z)$ where

$$(19) \qquad w_{f_2}(z) = w_{f_1}(z) \int \exp\left(-\int a dz\right) \frac{dz}{w_{f_2}^2(z)}.$$

Since

$$a(z) - \frac{p_0}{z} = \frac{p(z) - p_0}{z}$$

is free of singularity at $z = 0$,

$$\int a dz - p_0 \ln z$$

is also free of singularity at $z = 0$.

Suppose that $w_0 \neq 0$ in (5). Then

$$(20) \qquad w_{f_2}(z) = w_{f_1}(z) \int \frac{g(z) dz}{z^{p_0 + 2\alpha}}$$

where $g(z)$ is analytic and free of singularity at $z = 0$. Hence

$$(21) \qquad g(z) = \sum_{k=0}^{\infty} g_k z^k.$$

If α and α – *n* (where *n* is a non-negative integer) are the roots of the indicial equation

$$p_0 - 1 = -2\alpha + n$$

i.e. $p_0 + 2\alpha = n + 1$, then by inserting (21) into (20) we have

$$w_{f_2}(z) = w_{f_1}(z) \sum_{k=0}^{\infty} g_k \int z^{k-n-1}\, dz$$

i.e.

$$w_{f_2}(z) = w_{f_1}(z) \left\{ \sum_{k \neq 0}^{\infty} g_k \frac{z^{k-n}}{k-n} + g_n \log z \right\}. \tag{22}$$

When the indicial equation has double root, (22) becomes by putting $n = 0$,

$$w_{f_2}(z) = w_{f_1}(z) \left\{ \sum_{k=1}^{\infty} g_k \frac{z^{k}}{k} + g_0 \log z \right\}. \tag{23}$$

It now follows that $g(z)$ in (21) becomes

$$g(z) = \frac{z^{p_0 + 2\alpha} \exp\left(-\int a(z)\, dz\right)}{w_{f_1}^2(z)} \tag{24}$$

Suppose that $z = 0$ is an ordinary point of (2). In this case $w(z)$ is analytic and free of singularity at $z = 0$. Taking $\alpha = 0 = p_0 = q_0 = q_1$, (15) is automatically satisfied and (14) reduces to

$$k\,(k-1) w_k = -\sum_{m=0}^{k-1} (m p_{k-m} + q_{k-m}) w_m,\ (k \geq 1). \tag{25}$$

The case $k = 1$ is trivial.

We have

$$-2w_2 = q_2 w_0 + p_1 w_1, \qquad k = 2$$

$$-3.2w_3 = q_3 w_0 + (p_2 + q_2) w_1 + 2p_1 w_2, \qquad k = 3$$

$$-4.3w_4 = q_4 w_0 + (p_3 + q_3) w_1 + (2p_2 + q_2) w_2 + 3p_1 w_3, \qquad k = 4.$$

and so on. It can be easily checked that by successive substitutions all the coefficients w_k can be expressed in terms of w_0 and w_1. It now follows that w_0 and w_1 are the two arbitrary constants in the general solution of (2).

13.6 SOLUTIONS AT INFINITY

In order to determine the nature of solutions of (2) for large values of $|z|$, we make the variable transformation $z = \dfrac{1}{u}$. Since

$$\frac{dw}{dz} = -u^2 \frac{dw}{du};$$

$$\frac{d^2w}{dz^2} = 2u^3 \frac{dw}{du} + u^4 \frac{d^2w}{du^2}$$

then (2) takes the form

$$(26) \qquad \frac{d^2w}{du^2} + \left[\frac{2}{u} - \frac{1}{u^2}\,a\left(\frac{1}{u}\right)\right]\frac{dw}{du} + \frac{1}{u^4}\,b\left(\frac{1}{u}\right)w = 0.$$

We say that the point at infinity ($z = \infty$) is an ordinary point of (26). In other words, we say that the point at infinity ($z = \infty$) in the z-plane is an ordinary point if $2z - z^2\,a(z)$ and $z^4\,b(z)$ are analytic and free of singularities at $z = \infty$. This is expressed as

$$(27) \qquad a(z) = \sum_{k=0}^{\infty} a_{-k}\,z^{-k}$$

$$b(z) = \sum_{k=0}^{\infty} b_{-k}\,z^{-k}.$$

We say that the point at infiniy ($z = \infty$) is a regular singularity if $u = 0$ is a regular singularity of (26). In other words, we say that the point $z = \infty$ is a **regular singularity** of (2) if $za\,(z)$ and $z^2\,a(z)$ are analytic and free of singularities at $z = \infty$. This is expressed by putting $a_0 = b_0 = b_{-1} = 0$ in (27). In order to get a solution of (2) valid for large $|\,z\,|$ we make a series substitution of the form

$$(28) \qquad w = \frac{1}{z^{\alpha}} \sum_{k=0}^{\infty} w_{-k}\,z^{-k} \text{ in (2).}$$

Then an indicial equation for α is obtained and a second solution is correspondingly found from the second root of this equation provided the roots do not differ by a non-negative integer. In other cases, the second solution is obtained as before by using the first solution.

13.7 THE HYPERGEOMETRIC DIFFERENTIAL EQUATION

In practice, most of the second-order linear homogeneous differential equations possess three regular singularities. We can place the three singularities at prescribed points and it is not difficult to choose them at 0, 1 and ∞. We have in mind the **hypergeometric differential equation** which has regular singularities with exponents α_1 and α_2 of values 0, $1 - c$ at $z = 0$; 0, $c - a - b$ at $z = 1$, a, b at $z = \infty$. This equation is

$$(29) \qquad z(1 - z)\,w'' + \{c - (a + b + 1)z]w' - abw = 0.$$

Here $\quad p(z) = za\,(z) = \dfrac{c - (a + b + 1)\,z}{1 - z} = \displaystyle\sum_{k=0}^{\infty} p_k\,z^k$

with $\quad p_0 = c$ and $p_k = c - (a + b + 1)$ for $k > 0$,

$$q\,(z) = z^2\,b\,(z) = -\frac{ab}{1 - z}\,z = \sum_{k=0}^{\infty} q_k\,z^k$$

with $\quad q_0 = 0$ and $q_k = ab$ for $k > 0$.

Thus the indicial equation in (15) reduces to

$$\varphi\,(\alpha) = \alpha^2 + (c - 1)\,\alpha = \alpha\,(\alpha + c - 1)$$

and (16) takes the form $(k + \alpha)\,(k + \alpha + c - 1)\,w_k$

$$= -\sum_{v=0}^{k-1} \{(m - \alpha)\,[c - (a + b + 1)] - ab\}\,w_v$$

where $(k + 1 + \alpha)\,(k + \alpha + c)\,w_{k+1} - (k + \alpha)\,(k + \alpha + c - 1)\,w_k$

$$= \{ab - (k + \alpha)\,[c - (a + b + 1)]\}\,w_k$$

and

(30) $$(k + 1 + \alpha)\,(k + c + a)\,w_{k+1} = (k + a + \alpha)\,(k + b + \alpha)\,w_k.$$

In particular, we have

$$(\alpha + 1)\,(\alpha + c)\,w_1 = (\alpha + a)\,(\alpha + b)\,w_0$$
$$(\alpha + 2)\,(\alpha + c + 1)\,w_2 = (\alpha + a + 1)\,(\alpha + b + 1)\,w_1$$
$$(\alpha + 3)\,(\alpha + c + 2)\,w_2 = (\alpha + a + 2)\,(\alpha + b + 2)\,w_2$$

and so on. By successive substitutions we have for $k \geq 1$,

(31) $$w_k = \frac{(\alpha + a + k + 1)\ldots(\alpha + a)\,(\alpha + b + k + 1)\ldots(\alpha + b)}{(\alpha + c + k - 1)\ldots(\alpha + c)\,(\alpha + k)\ldots(\alpha + 1)}\,w_0.$$

i.e.

(32) $$w_k = \frac{v\,(\alpha + a + k)\,v\,(\alpha + b + k)}{v\,(\alpha + c + k)\,v\,(\alpha + k + 1)}\,w_0.$$

where $v\,(x + k) = x\,(x + 1)\ \ldots\ (x + k + 1).$

Furthermore, $v\,(x + k + 1) = (x + k)\,v\,(x + k)$

i.e. $v\,(y + 1) = yv\,(y)$

where $y = x + k$. It follows that $v\,(y)$ is proportional of $\Gamma\,(y)$. Hence (32) becomes

$$w_k = \frac{\Gamma(\alpha + a + k)\,\Gamma\,(\alpha + b + k)}{\Gamma\,(\alpha + c + k)\,\Gamma\,(\alpha + k + 1)}\,w_0.$$

The two roots of the indicial equation are 0 and $1 - c$. The solution of (29) corresponding to $\alpha = 0$ and $w_0 = \Gamma(c)/\Gamma(a)\,\Gamma(b)$ is called **hypergeometric function.** We denote by

(33) $$F\,(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a + k)\,\Gamma(b + k)}{k\,!\,\Gamma(c + k)}\,z^k.$$

This series converges for $|\,z\,| < 1$. If c is not an integer the other solution corresponding to $\alpha = 1 - c$ will be of the form

$$z^{1-c}\,F\,(1 + a - c,\ 1 + b - c,\ 2 - c,\ z).$$

Hence, the general solution of (29) is of the form

$$AF\,(a, b, c, z) + Bz^{1-c}\,F\,(1 + a - c,\ 1 + b - c,\ 2 - c,\ z).$$

13.8 SOME SIMPLE CONSEQUENCES OF THE FUNCTION *F* (*a*, *b*, *c*, *z*) IN (33)

We will obtain different types of functions for special choice of the parameters *a*, *b*, *c*.

(*i*) Setting $b = c$ in (33), we get

$$F(a, b, c, z) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(a)\, k!} z^k.$$

Since
$$\frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1) \ldots (a+k-1)$$
$$= (-1)^k (-a)(-a-1) \ldots (-a-k+1).$$

Hence
$$F(a, b, c, z) = \frac{1}{a(1-z)}.$$

(*ii*) Setting $a = b = 1$, $c = 2$ in (33), we get

$$F(1, 1, 2, z) = \sum_{k=0}^{\infty} \frac{z^k}{1+k} = -\frac{1}{z} \log(1-z).$$

(*iii*) Setting $c = b$ and $z = \frac{z}{a}$ in (33), we get

$$F\left(a, b, b, \frac{z}{a}\right) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{a^k\, \Gamma(a)} \frac{z^k}{k!}.$$

Since
$$\lim_{a \to \infty} \frac{\Gamma(a+k)}{a^k\, \Gamma(a)} = 1$$

hence
$$\lim_{a \to \infty} F\left(a, b, b, \frac{z}{a}\right) = e^z.$$

We can express the derivative of $F(a, b, c, z)$ as a hypergeometric function and leave to the reader to verify the following relation

$$(34) \qquad F'(a, b, c, z) = \frac{ab}{c} F(a+1, b+1, c+1, z).$$

It can be easily seen that the hypergeometric function

$$F(a, b, c, z) = F(b, a, c, z)$$

is symmetric in the first two parameters. Also, if b is a negative integer, say $-m$, then $\frac{\Gamma(b+k)}{\Gamma(b)}$ will be zero for $k > m$; otherwise it is evaluated from the residues of $\Gamma(z)$ and thus (33) becomes a polynomial.

$$(35) \qquad F(a, -m, c, z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{k=0}^{m} \frac{\Gamma(a+k)}{\Gamma(c+k)} \frac{m!(-1)^k z^k}{k!(m-k)!}$$

Setting $a = m + q$ in (35), we get

$$(36) \qquad F(m+q, -m, c, z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma(a+k)}{\Gamma(c+k)} (-z)^k.$$

This is called a **Jacobi polynomial** and is denoted by $J_m(q, c, z)$.

13.9 BESSEL'S DIFFERENTIAL EQUATION

Bessel's differential equation of order n is given by

$$(37) \qquad z^2 w'' + z w' + (z^2 - n^2) w = 0.$$

If $n \geq 0$, a solution of the equation (37) is

$$(38) \quad J_n(z) = \frac{z^n}{2^n \Gamma(n+1)} \left\{ 1 - \frac{z^2}{2(2n+2)} + \frac{z^4}{2.4\,(2n+2)(2n+4)} - \dots \right\}$$

$J_n(z)$ is called **Bessel's function** of the first kind of order n.

If $n \neq$ an integer, the general solution of (37) is given by

$$(39) \qquad w = AJ_n(z) + BJ_{-n}(z)$$

where A and B are arbitrary constants.

We list some useful properties of Bessel functions and leave the verification to the reader

$$(i) \qquad \exp\left\{\frac{1}{2} z \left(t - \frac{1}{t}\right)\right\} = \sum_{n=-\infty}^{\infty} J_n(z)\, t^n .$$

This is called the **generating function** for the Bessel functions of the first kind for integer values of n. Note that if n is an integer then $J_{-n}(z) = (-1)^n J_n(z)$ and (38) fails to give the general solution. By taking the series of the form $(\ln z) \sum_{k=0}^{\infty} a_k z^k$, the general solution in this case can be obtained.

(*ii*) **Recursion Formula:**

$$z J_{n-1}(z) - 2n J_n(z) + z J_{n+1}(z) = 0$$

$$(iii) \qquad J_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - z \sin\theta)\, d\theta ,$$

where n is an integer.

$$(iv) \qquad J_n(z) = \frac{1}{2\pi i} \int_{\gamma} t^{-n-1} \exp\left\{\frac{1}{2} z \left(t - \frac{1}{t}\right)\right\} dt, \; n = 0, \pm 1, \pm 2, \pm 3, \dots,$$

where γ is any simple closed curve enclosing $t = 0$.

$$(v) \qquad J_n(z) = \frac{z^n}{1.3.5 \dots (2n-1)\pi} \int_{-1}^{1} e^{izt} (1 - t^2)^{n - \frac{1}{2}}\, dt .$$

If n is a positive integer, a second solution to Bessel's differential equation (37) is given by

$$(40) \quad S_n(z) = J_n(z) \ln z - \frac{1}{2} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n}$$
$$- \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)(n+k)!} \left(\frac{z}{2}\right)^{2k+n} \{\rho(k) + \rho(n+k)\}$$

where $\rho(k) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}$ and $\rho(0) = 0$.

This is called Bessel's function of the second kind of order n or **Neumann's function.** For a full discussion of the function of the second kind one may refer "A treatise on the theory of Bessel functions" by G.N. Watson [14].

13.10 LEGENDRE'S DIFFERENTIAL EQUATION

Legendre's differential equation of order n is given by

(41) $(1 - z^2)\, w'' - 2zw' + n\,(n + 1)\, w = 0.$

The general solution of the equation is

$$(42)\quad w = A\left\{1 - \frac{n\,(n+1)}{2!}z^2 + \frac{n\,(n-2)\,(n+1)\,(n+3)}{4!}z^4 - \ldots\right\}$$

$$+ B\left\{z - \frac{(n-1)\,(n+2)}{3!}z^3 + \frac{(n-1)\,(n-3)\,(n+2)\,(n+4)}{5!}z^5 - \ldots\right\}$$

If n is zero or a positive integer, polynomial solutions of degree n are obtained and these polynomial solutions are called **Legendre polynomials.** We also find that they can be expressed by **Rodrigues' formula**

$$(43)\qquad P_n(z) = \frac{1}{2^n\, n!}\frac{d^n}{dz^n}(z^2 - 1)^n.$$

We list some useful properties of Legendre polynomials and leave the verification to the reader.

$$(i)\qquad \frac{1}{\sqrt{1 - 2zt + t^2}} = \sum_{n=0}^{\infty} P_n(z)\, t^n$$

This is called the generating function for Legendre polynomials.

(*ii*)

$$P_n(z) = \frac{(2n)!}{2^n\,(n!)^2}\left\{z^n - \frac{n(n-1)}{2(2n-1)}z^{n-2} + \frac{n(n-1)\,(n-2)\,(n-3)}{2.4(2n-1)\,(2n-3)}z^{n-4} - \ldots\right\}$$

Recursion formula:

(*iii*) $(n + 1)\, P_{n+1}(z) - (2n + 1)\, z\, P_n(z) + n\, P_{n-1}(z) = 0$

(*iv*) $P_n(z) = \frac{1}{2\pi i}\int_\gamma \frac{(t^2 - 1)^n}{2^n (t - z)^{n+1}}\, dt$

where γ is any simple closed curve enclosing the pole $t = z$.

$$(v)\quad \int_{-1}^{1} P_m(z)\, P_n(z)\, dz = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2n+1} & \text{if } m = n \end{cases}$$

If n is not a positive integer, there are two infinite series solutions obtained from (42). These solutions to Legendre's equation are called **Legendre functions.**

EXERCISES

1. Show that the series

$$(i)\ \sum_{n=0}^{\infty} \frac{z^n}{2n+1} \qquad \text{and} \qquad (ii)\ \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$$

are analytic continuations of each other.

2. Prove that $f_1(z) = \frac{1}{1+i} \sum_{n=0}^{\infty} \left(\frac{z+i}{1+i}\right)^n$ is an analytic continuation of $f_2(z) = \sum_{n=0}^{\infty} z^n$, showing the regions of convergence of the series.

3. Let $\quad f(z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{3^n}$

Find an analytic continuation of $f(z)$ which converges for $z = 3 - 4i$.

4. Prove that the series

$$z^{1!} + z^{2!} + z^{3!} + \ldots$$

has the natural boundary $|z| = 1$.

5. By use of analytic continuation, show that

$$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}.$$

Legendre Functions

6. Prove that

$$(i)\ \int_{-1}^{1} P_m(z)\, P_n(z)\, dz = 0 \qquad \text{if } m \neq n$$

$$(ii)\ \int_{-1}^{1} P_m(z)\, P_n(z)\, dz = \frac{2}{2n+1} \qquad \text{if } m = n.$$

7. Establish the following recurrence relations for the Legendre polynomials

(*i*) $nP_n - (2n-1)\, zP_{n+1} + (n-1)\, P_{n-2} = 0$

(*ii*) $zP'_n - P'_{n-1} = nP_n$

(*iii*) $P'_n - zP'_{n-1} = nP_{n-1}$

(*iv*) $P'_{n+1} - P'_{n-1} = (2n+1)\, P_n$

(*v*) $(z^2 - 1)\, P'_n = nzP_n - nP_{n-1}$

8. Prove that

$$P_{2n}(0) = (-1)^n \left(\frac{2n!}{2^{2n}\,(n!)^2}\right)$$

$$P_{2n+1}(0) = 0$$

9. Prove that $\int_{-1}^{1} \frac{P_n(x)}{\sqrt{1-x^2}}\, dx = \pi\, [P_n(0)]^2$

Bessel Functions

10. Show that

(*i*) $J_0'(z) = -J_1(z)$

(*ii*) $\int z^3 J_2(z)\, dz = z^3 J_3(z) + C$, C a constant

(*iii*) $\int z^3 J_0(z)\, dz = z^3 J_1(z) - 2z^2 J_2(z) + C$

(*iv*) $J_{1/2}(z) = \sqrt{2}/\pi z \sin z$

(*v*) $J_{-1/2}(z) = \sqrt{2}/\pi z \cos z$

11. Prove that

(*i*) $J_{3/2}(z) \sin z - J_{-3/2}(z) \cos z = \sqrt{2}/\pi z^3$

(*ii*) $J_n'(z) = \frac{1}{2}\{J_{n-1}(z) - J_{n+1}(z)\}$

(*iii*) If Re $(z) > 0$, then

$$\int_0^\infty e^{-zt} J_0(t)\, dt = \frac{1}{\sqrt{z^2+1}}$$

12. Establish the following recurrence relations for the functions.

(*i*) $z(J_{v-1} + J_{v+1}) = 2vJ_v$

(*ii*) $vJ_v = z(J_{v-1} - J_v')$

(*iii*) $2J_v' = J_{v-1} - J_{v+1}$

(*iv*) $vJ_v = z(J_{v+1} + J_v')$

13. Prove that $zJ_{n-1}(z) - 2nJ_n(z) + zJ_{n+1}(z) = 0$

Hypergeometric Functions

14. Show that

(*i*) $F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, z^2\right) = \frac{\sin^{-1} z}{z}$

(*ii*) $zF(1, 1, 2, -z) = \ln(1+z)$

(*iii*) $F\left(\frac{1}{2}, 1, \frac{3}{2}, -z^2\right) = \frac{\tan^{-1} z}{z}$

15. Prove that $\frac{d}{dz} F(a, b, c, z) = \frac{ab}{c} F(a+1, b+1, c+1, z)$.

16. By using power series solve each of the following differential equations. Find also the region of convergence.

(*i*) $Y'' + 2Y' + Y = 0$

(*ii*) $Y'' + 2Y = 0$

(*iii*) $zY'' + 2Y' + zY = 0$

(*iv*) $(1 - z^2) Y'' + 2Y = 0$

17. Solve by power series method

$$Y'' + z^2 Y = 0$$

subject to the conditions $Y(0) = 1$, $Y'(0) = 1$. Find also the region of convergence.

18. Solve by power series method

$$z Y'' + (1 + z) Y' - Y = 0$$

and determine the region of convergence.

19. Show that the differential equation

$$Y'' - zY' + nY = 0$$

has two independent series solutions representing entire functions given by

$$Y = 1 - n + \frac{z^2}{2\,!} + \frac{n(n-2)}{4\,!} z^4 - \frac{n(n-2)\,(n-4)}{6\,!} z^6 + \ldots$$

$$Y = z - \frac{(n-1)}{3\,!} z^3 + \frac{(n-1)\,(n-3)}{5\,!} z^5 - \frac{(n-1)\,(n-3)\,(n-5)}{7\,!} z^7 + \ldots$$

14

APPROXIMATION BY RATIONAL FUNCTIONS AND POLYNOMIALS

14.1 UNIFORM APPROXIMATION

We begin this chapter by recalling some useful and fundamental theorems on **approximation.** We assume that the reader is familiar with the **Weierstrass theorem** on approximation and it is given in almost all standard books on Real Analysis. For completeness, we first state and prove the Weierstrass theorem and then discuss the approximation by rational functions and polynomials in the complex setting.

We know that for many applications it is convenient to **"approximate"** continuous functions by functions of an elementary nature. The most reasonable definition that one can use to make the word "approximate" is that every point of the given domain the approximating function shall not differ from the given function by more than the preassigned error. The above *definition* is also referred to as **uniform approximation** and is connected with uniform convergence.

Theorem 1. *Let* $A = [a, b] \subseteq R$ *be compact. Suppose that* f *is continuous on* A*. Then* f *can be uniformly approximated on* A *by continuous piecewise linear functions.*

Proof: Since A is a compact set hence f is uniformly continuous on A. Therefore, given $\varepsilon > 0$, we divide $A = [a, b]$ into subintervals by the points c_k, $k = 0, 1, 2, ..., n$, with $a = c_0 < c_1 < c_2 < ... < c_n = b$ so that $c_k - c_{k-1} < \delta(\varepsilon)$. Connecting the points $(c_k, f(c_k))$ by line segments, and defining the resulting continuous piecewise linear function by φ_ε, we see that φ_ε approximates f uniformly on A within ε. For illustration see Fig. 14.I.

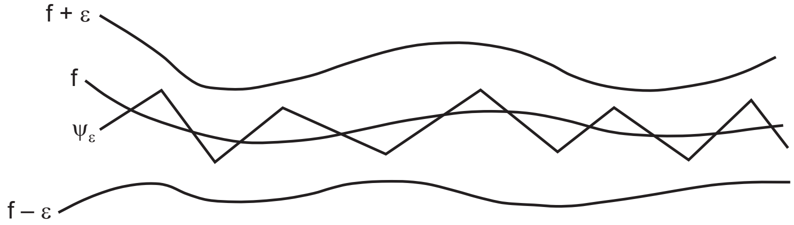


Fig. 14.I

We now discuss in detail the **Bernstein theorem.** This theorem gives us a constructive method of finding a sequence of polynomials which converges uniformly on [0, 1] to the given continuous function. By using Bernstein theorem, we will prove Weierstrass theorem on approximation.

Definition: Let $f: [0, 1] \to R$. We define the nth Bernstein polynomial for f as

$$(1) \qquad B_n(x) = B_n(x; f) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

Observe that the value $B_n(x; f)$ of the polynomial at the point x is calculated from the values $f(0)$, $f\left(\frac{1}{n}\right)$, $f\left(\frac{2}{n}\right)$, …, $f(1)$ with certain non-negative weight factors

$$g_k(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

It can be easily checked that g_k takes its maximum value at the point $\frac{k}{n}$. Write

$$(2) \quad (u+t)^n = \sum_{k=0}^{n} \binom{k}{n} u^k t^{n-k},$$

where $\binom{n}{k}$ denotes the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Note that

$$(3) \quad \binom{k-1}{n-1} = \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{k}{n}\binom{n}{k},$$

$$(4) \quad \binom{k-2}{n-2} = \frac{(n-2)!}{(k-2)!(n-k)!} = \frac{k(k-1)}{n(n-1)}\binom{n}{k}.$$

Setting $u = x$ and $t = 1 - x$ in (2), we get

$$(5) \quad 1 = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}.$$

By replacing n by $n - 1$ and k by j in (5), we have

$$(6) \quad 1 = \sum_{j=0}^{n-1} \binom{n-1}{j} x^j (1-x)^{n-1-j}.$$

Multiplying both sides of (6) by x and applying the identity (3), we get

$$(7) \quad x = \sum_{j=0}^{n-1} \frac{j+1}{n}\binom{n}{j+1} x^{j+1} (1-x)^{n-(j+1)}.$$

Setting $k = j + 1$ in the right side of (7), we have

$$(8) \quad x = \sum_{k=1}^{n} \frac{k}{n}\binom{n}{k} x^k (1-x)^{n-k}.$$

Note that (8) can be written

$$(9) \quad x = \sum_{k=0}^{n} \frac{k}{n}\binom{n}{k} x^k (1-x)^{n-k}.$$

Replacing n by $n - 2$ in (5) and using the identity (4) and by a similar calculation as above, we get

$$(n^2 - n)\, x^2 = \sum_{k=0}^{n} (k^2 - k)\binom{n}{k} x^k (1-x)^{n-k}.$$

Hence, we have

$$(10) \quad \left(1 - \frac{1}{n}\right)x^2 + \frac{1}{n}x = \sum_{k=0}^{n} \left(\frac{k}{n}\right)^2 \binom{k}{n} x^k (1-x)^{n-k}.$$

Multiplying (5) by x^2, (6) by $- 2x$, and adding them to (7), we get

$$(11) \quad \left(\frac{1}{n}\right)x\,(1 - x) = \sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^n \binom{n}{k} x^k (1-k)^{n-k}.$$

It follows from (5) that $B_n(x; f_0) = f_0(x) = 1$, from (8) that $B_n(x; f_1) = f_1(x) = x$. We also find from (10) that the nth Bernstein polynomial for the function $f_2(x) = x^2$ is

$$B_n(x, f_2) = \left(1 - \frac{1}{n}\right)x^2 + \frac{1}{n}x$$

which converges uniformly on [0, 1] to f_2.

We now prove Bernstein theorem on approximation.

Theorem 2. *Let $f : [0, 1] \to R$ be continuous. Let the sequence of Bernstein polynomials for f be defined by*

$$B_n(x) = B_n(x\,; f) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

Then $B_n(x)$ converges uniformly on [0, 1] to f.

Proof : Multiplying (5) by $f(x)$, we get

$$f(x) = \sum_{k=0}^{n} f(x)\binom{n}{k} x^k (1-x)^{n-k}.$$

Now $$f(x) - B_n(x) = \sum_{k=0}^{n} \left\{ f(x) - f\left(\frac{k}{n}\right)\right\} \binom{n}{k} x^k (1-x)^{n-k}.$$

Hence

$$(12) \qquad | f(x) - B_n(x) | \le \sum_{k=0}^{n} \left| f(x) - f\left(\frac{k}{n}\right)\right| \binom{n}{k} x^k (1-x)^{n-k}.$$

Note the following facts. If k is such that $\frac{k}{n}$ is near x, then the corresponding term in the sum (12) is small, since f is continuous at x. If $\frac{k}{n}$ is not near x, the factor involving f is less than $2M$ where M is the bound for f. In order to estimate the expression in (12) we consider those values of k for which $x - \frac{k}{n}$ is small and those for which $x - \frac{k}{n}$ is large.

Let $\varepsilon > 0$. Choose n so large that

(13) $\quad n \leq \sup\,\{(\delta(\varepsilon))^{-4}, M^2/\varepsilon^2\}$,

and break (12) into two sums. We first consider the sum taken over those k for which

$$\left| x - \frac{k}{n} \right| < n^{-1/4} \leq \delta(\varepsilon).$$

Hence $$\sum_k \varepsilon \binom{n}{k} x^k (1-x)^{n-k} \leq \varepsilon \sum_{k=1}^{n} \binom{n}{k} x^k (1-x)^{n-k} = \varepsilon.$$

We now consider the sum taken over those k for which

$$\left| x - \frac{k}{n} \right| \geq n^{-1/4}$$

i.e. $$\left(x - \frac{k}{n} \right)^2 \geq n^{-1/2}.$$

By using (11) we find that

$$\begin{aligned}
&\sum_k 2M \binom{n}{k} x^k (1-x)^{n-k} \\
&= 2M \sum_k . \frac{\left(x - \frac{k}{n} \right)^2}{\left(x - \frac{k}{n} \right)^2} \binom{n}{k} x^k (1-x)^{n-k} \\
&\leq 2M\sqrt{n} \sum_{k=1}^{n} \left(x - \frac{k}{n} \right)^2 \binom{n}{k} x^k (1-x)^{n-k} \\
&\leq 2M\sqrt{n} \left\{ \frac{1}{n} x\,(1-x) \right\} \\
&\leq \frac{M}{2\sqrt{n}}, \text{ since } x\,(1-x) \leq \frac{1}{4}
\end{aligned}$$

on [0, 1]. Hence, for n chosen in (13) we have

$$|f(x) - B_n(x)| < 2\varepsilon.$$

Thus the theorem is proved.

We will see below that Weierstrass approximation theorem is a direct consequence of Bernstein theorem.

Theorem 3. *Let $f : [a, b] \to R$ be continuous. Then f can be uniformly approximated by polynomials.*

Proof: Let f be defined on $[a, b]$. Define another function g on [0, 1] by $g(t) = f((b - a)\, t + a)$ where $t \in [0, 1]$. The function g is continuous and hence can be approximated by Bernstein polynomials. By changing the variable we get a polynomial approximation to the function f.

Suppose that $f(z)$ is analytic in $|z| \leq \rho$ and let $p(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial. By using the Taylor's theorem for analytic function it can be seen that given an $\varepsilon > 0$, $|f(z) - p_n(z)| \leq \varepsilon$ for $|z| \leq \rho$.

If $f(z)$ is not analytic, we cannot expand $f(z)$ in power series. But Weierstrass theorem asserts that we can approximate functions which are merely continuous.

Approximation of analytic functions by analytic functions is totally different. Let Ω be a region bounded by a simple closed curve γ. Suppose that $f(z)$ and $p(z)$ are two functions analytic in Ω and on γ. Suppose also that $|f(z) - p(z)| \leq \varepsilon$ on γ. By maximum principle, this inequality and hence approximation persists throughout Ω. It follows from Cauchy's inequality that

$$(14) \quad |f^{(n)}(z) - p^{(n)}(z)| \leq \frac{n!\, L(\gamma)}{2\pi\, \delta^{n+1}}\, \varepsilon$$

where z belongs to a point set, say $E \subset \Omega$ and the distance of those points from γ is no less than δ. In (14) we fix E and n and allow $\varepsilon \to 0$. It then follows from (14) that the nth derivative of the approximant is also an approximation to the nth derivative of the approximate. Thus we see that the approximation over regions carries with it the simultaneous approximation of all the derivatives.

Suppose that f is continuous on $[a, b]$. We know that we can approximate f by a polynomial. The following question naturally arises. Let $x_1, x_2, \ldots, x_n$ be n points of $[a, b]$ and let $\varepsilon > 0$ be given. Can we find a polynomial $p(x)$ such that

$$|f(x) - p(x)| \leq \varepsilon, x \in [a, b]$$

and

$$p(x_i) = f(x_i), i = 1, 2, \ldots, n\, ?$$

In the complex setting the following theorem of Walsh answers this question.

Theorem 4. *Let E be a point set which is closed and bounded in the complex plane C. Let $z_1, z_2, \ldots, z_n$ be n distinct points of E. Suppose that f (z) is defined on E and is approximable by polynomials there. Then f(z) is approximable by polynomials p and satisfy the property*

$$p(z_i) = f(z_i), i = 1, 2, \ldots, n.$$

Proof : Let $\varepsilon > 0$ be given. Choose a polynomial $p(z)$ such that

$$|f(z) - p(z)| \le \varepsilon, z \in E.$$

Define

$$q(z) = \sum_{k=1}^{n} (f(z_k) - p(z_k))\, u_k(z)$$

where $$u_k(z) = \frac{v(z)}{(z - z_k)v'(z_k)}$$

and $$v(z) = \prod_{k=1}^{n} (z - z_k).$$

It follows that

$$q(z_k) = f(z_k) - p(z_k), k = 1, 2, ..., n.$$

Write $$M = \sum_{k=1}^{n} \max_{z \in E} |u_k(z)|.$$

Now $$\max_{z \in E} |q(z)| \le \sum_{k=1}^{n} |f(z_k) - p(z_k)| \max_{z \in E} |u_k(z)| \le \varepsilon M.$$

Observe that M depends on $z_1, z_2, ..., z_n$ and E only.

Set $$p_1(z) = p(z) + q(z)$$

Then $$p_1(z_k) = p(z_k) + q(z_k) = f(z_k), k = 1, 2, ..., n.$$

Further

$$\begin{aligned} |f(z) - p_1(z)| &\le |f(z) - p(z)| + |q(z)| \\ &\le \varepsilon + M\varepsilon, z \in E. \end{aligned}$$

Example: Suppose that $f(z)$ is analytic in $|z| \le \rho$. Since $f(z)$ is expressed in terms of power series which is uniformly convergent for $|z| \le \rho$. Hence $f(z)$ is approximable there by polynomials. Let $z_1, z_2, ..., z_n$ be distinct points in $|z| \le \rho$. Then we can find a polynomial $p(z)$ such that

$$|f(z) - p(z)| \le \varepsilon, |z| \le \rho$$

and $$p(z_k) = f(z_k), k = 1, 2, ..., n.$$

We say that the power series representation of analytic functions provides us with a very specific procedure of approximation by polynomials, but this process can be applied in a circular disk contained in the set in which the given function is analytic. We will now discuss **Runge's theorem** which provides a **global approximation** by rational functions. We will present here the simple version of Runge's theorem and refer to [8] for generalized version of the theorem. Initially, we need some new definitions.

14.2 LOCALLY ANALYTIC FUNCTIONS

Let E be a set which is dense in itself. Let $f(z)$ be a single-valued function defined on E. We say that f is locally analytic on E if given any $z_0 \in E$, there is a neighbourhood N of z_0 and a power series

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

such that $$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

for all $z \in N \cap E$.

When E is a region, the concept of a locally analytic function on E reduces to that of an analytic function on a region.

Example: Suppose we divide the z-plane into infinitely many closed squares $\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_n, \ldots$ with sides of unit length parallel to the coordinate axes. Let $E = A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots$ be the set of interior points of these squares.

Then the function defined by

$$f(z) = z^n, z \in A_n,$$

is locally analytic on E.

Let $E = \Omega$ be an arbitrary open set. The set Ω is a region if it is connected, but otherwise Ω is the union of countably many **disjoint regions**. Suppose we divide the z-plane into infinitely many closed squares with sides of length $\frac{1}{3^n}$ parallel to the coordinate axes where n is any positive integer. Here the origin is a vertex of one of the squares. Let $\overline{A}$ be any of these squares such that $\overline{A}$ and the eight squares bordering $\overline{A}$ are all contained in Ω as shown in the Fig. 14.II. Then the union of all such squares minus its boundary is an open set Ω'_n contained in Ω.

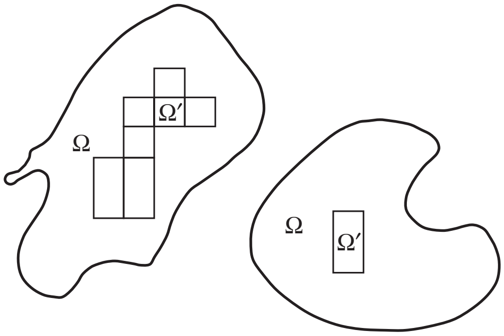


Fig. 14.II

If Ω_n is the intersection of Ω'_n with the open square

$$-3^n < x < 3^n, -3^n < y < 3^n$$

then Ω_n is a bounded open set such that $\overline{\Omega}_n \subset \Omega$.

Recall that a **Jordan curve** is a **homeomorphic image** of the unit circle. We will use some facts of Jordan curves without discussing the properties. In the above example, the boundary of Ω_n consists of a finite number of closed rectifiable Jordan curves, each of which is made up of a finite number of line segments. Moreover, $\overline{\Omega}_n \subset \Omega_{n+1}$ and given any compact set $F \subset \Omega$, we can find an integer $N(F) > 0$ such that $F \subset \Omega_n$ for all $n > N(F)$. We summarize these properties by saying that $\{\Omega_n\}$ is an increasing sequence of open sets approximating Ω.

We now prove a theorem on approximation by rational functions.

Theorem 5. *Let Ω be any open set. Suppose that $f(z)$ is locally analytic on Ω. Then there exists a sequence of rational functions $\{r_n(z)\}$ converging uniformly to $f(z)$ inside Ω.*

Proof: Let $\{\Omega_n\}$ be the increasing sequence of open sets approximating Ω as mentioned above. Let γ_n denotes the boundary of Ω_n. Then

$$(15) \qquad f(z) = \frac{1}{2\pi i}\int_{\gamma_{n+1}} \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

for all $z \in \overline{\Omega}_n$,

where the integral along γ_{n+1} is the sum of integrals along the separate closed rectifiable Jordan curves forming γ_{n+1}.

Since $f(z)$ is continuous on γ_{n+1}, it follows that

$$|f(\zeta') - f(\zeta'')| < \varepsilon \text{ where}$$

$$|\zeta' - \zeta''| < \delta(\varepsilon'),\ \zeta', \zeta'' \in \gamma_{n+1}.$$

Divide all the curves making up γ_{n+1} into arcs Γ_k, $k = 1, 2, ..., N(\varepsilon')$. Let $\zeta_k'^{(n+1)}$ be the initial and $\zeta_k''^{(n+1)}$ the final point of Γ_k. Let l_k be the length of Γ_k where

$$l_k < \min\{\delta(\varepsilon'), \varepsilon\}$$

for all $\qquad k = 1, 2, ..., N(\varepsilon')$.

Observe that the final point of the arc Γ_k coincides with the initial point of the arc Γ_{n+1}, except for those k for which Γ_k and Γ_{k+1} lie on different curves making up γ_{n+1}.

We now approximate the integral (15) by the sum

$$(16) \qquad S^{(n+1)}(z) = \frac{1}{2\pi i}\sum_{k=1}^{N(\varepsilon')} \frac{f(\zeta_k'^{(n+1)})}{\zeta_k'^{(n+1)} - z}(\zeta_k''^{(n+1)} - \zeta_k'^{(n+1)}).$$

We have

$$|f(z) - S^{(n+1)}(z)|$$

$$= \left| \frac{1}{2\pi i}\int_{\gamma_{n+1}} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\sum_{k=1}^{N(\varepsilon')} \frac{f(\zeta_k'^{(n+1)})}{\zeta_k'^{(n+1)} - z}(\zeta_k'^{(n+1)} - \zeta_k''^{(n+1)}) \right|$$

$$= \left| \frac{1}{2\pi i}\sum_{k=1}^{N(\varepsilon')} \int_{\Gamma_k} \left[\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta_k'^{(n+1)})}{\zeta_k'^{(n+1)} - z} \right] d\zeta \right|$$

$$= \left| \frac{1}{2\pi i} \sum_{k=1}^{N(\varepsilon')} \int_{\Gamma_k} \left[\frac{f(\zeta) - f(\zeta'^{(n+1)}_k)}{\zeta - z} + f(\zeta'^{(n+1)}_k) \frac{\zeta'^{(n+1)}_k - \zeta}{(\zeta - z)(\zeta'^{(n+1)}_k - z)} \right] d\zeta \right|.$$

Set $$M_{n+1} = \max_{z \in \gamma_{n+1}} |f(z)|$$

and let d_{n+1} be the distance between $\overline{\Omega}_n$ and γ_{n+1}. Then

$$|f(z) - S^{(n+1)}(z)|$$

$$\leq \frac{1}{2\pi} \sum_{k=1}^{N(\varepsilon')} \int_{\Gamma_k} \left| \frac{f(\zeta'^{(n+1)}_k)}{\zeta - z} \right| d\Gamma$$

$$+ \frac{1}{2\pi} \sum_{k=1}^{N(\varepsilon')} \int_{\Gamma_k} |f(\zeta'^{(n+1)}_k)| \frac{|\zeta'^{(n+1)}_k - \zeta|}{|(\zeta - z)(\zeta'^{(n+1)}_k - z)|} d\Gamma$$

$$(17) \qquad < \frac{1}{2\pi} \sum_{k=1}^{N(\varepsilon')} \frac{\varepsilon l_k}{d_{n+1}} + \frac{1}{2\pi} \sum_{k=1}^{N(\varepsilon')} M_{n+1} \frac{\varepsilon l_k}{d_{n+1}^2}.$$

The right side of (17) can be made less than any preassigned $\varepsilon > 0$ by choosing ε sufficient small. Put $\varepsilon = \varepsilon_n$ where $\varepsilon_n \to 0$ as $n \to \infty$.

We thus obtain a sequence of rational functions $\{S^{(n+1)}(z)\}$ which converges uniformly to $f(z)$ on every closed set $\overline{\Omega}_1, \overline{\Omega}_2, \ldots$ and hence uniformly to $f(z)$ inside Ω. The proof is complete by choosing $\{r_n(z)\} = \{S^{(n+1)}(z)\}$.

In order to prove Runge's theorem we need a lemma.

Lemma: *Let G be the complement of a compact set F and let* $\zeta \in G$*. Suppose that*

$$r(z) = \frac{P(z)}{(z-\zeta)^k},\ P(\zeta) \neq 0,$$

where P (z) is a polynomial whose degree does not exceed k. Let η *belongs to the same component of G as* ζ*. Then, given* $\varepsilon > 0$*, there exists a rational function*

$$r_1(z) = \frac{P_1(z)}{(z-\eta)^{k_1}},\ P_1(\eta) \neq 0,$$

such that $$|r_1(z) - r(z)| < \varepsilon,\ z \in F.$$

Here the degree of the polynomial $P_1(z)$ *does not exceed* k_1.

Proof: Let g be the component of G containing the points ζ and η. Let $\Gamma \subset g$ be a Jordan curve joining ζ and η and let ρ be the distance between Γ and F. Divide Γ into arcs $\tau_1, \tau_2, \ldots, \tau_m$ of diameter less than $\frac{1}{2}\rho$. Let $\zeta = \zeta_0, \zeta_1, \ldots, \zeta_m = \eta$ be the points of division. We construct the rational function

$$r_1(z) = \frac{P(z)}{(z-\zeta)^k}\left[1-\left(\frac{\zeta-\zeta_1}{z-\zeta_1}\right)^{n_1}\right]^k = \frac{P_2(z)}{(z-\zeta_1)^{n_1 k}},$$

where the degree of $P_2(z)$ does not exceed $n_1 k$. Let

$M = \max\limits_{z \in F} |f(z)|$. Then

$$|r_1(z) - r(z)| = |r(z)|\left|\left[1-\left(\frac{\zeta-\zeta_1}{z-\zeta_1}\right)^{n_1}\right]^k - 1\right|$$

$$\leq M\left[\binom{k}{1}\left(\frac{1}{2}\rho/\rho\right)^{n_1} + \ldots\right]$$

$$< M\frac{2^k}{2^{n_1}} \text{ for all } z \in F.$$

By choosing n_1 sufficiently large, we find that

$$|r_1(z) - r(z)| < \frac{\varepsilon}{m}, z \in F.$$

By repeating the argument, we obtain the required rational function at the mth step.

We now turn to the subject of approximation by polynomials and prove Runge's theorem.

Theorem 6. *Suppose that* Ω *is a union of countably disjoint simply connected regions which do not contain the point at infinity. Suppose also that f(z) is locally analytic on* Ω. *Then there exists a sequence of polynomials* $\{P_n(z)\}$ *which converges uniformly to f(z) inside* Ω.

Proof: Let $\{\varepsilon_n\}$ be a sequence of positive numbers which converge to zero. Suppose that $\{\Omega_n\}$ is an increasing sequence of open sets approximating the open set Ω. Suppose also that

$$|f(z) - S^{(n+1)}(z)| < \varepsilon_n \text{ for all } z \in \overline{\Omega}_n, \text{ where}$$

$$S^{(n+1)}(z) = \frac{1}{2\pi i}\sum_{k=1}^{N(n+1)} \frac{f(\zeta_k'^{(n+1)})}{\zeta_k'^{(n+1)} - z}(\zeta_k''^{(n+1)} - \zeta_k'^{(n+1)}). \tag{18}$$

Note that we are following the notation as in the proof of Theorem 5. Note also that each component of Ω_n is a simply connected region. Hence, if $|z| < \rho_n$ is a disk containing $\overline{\Omega}_n$ but not some point η, we can join η to every point of the boundary γ_{n+1} of the set Ω_{n+1}, see Fig. 14.III. Applying the Lemma to every rational function $S^{(n+1)}(z)$ in (18) and choosing $\varepsilon = \dfrac{\varepsilon_n}{N^{(n+1)}}$ we construct a rational function

$$Q_n(z) = \frac{P_1(z)}{(z-\eta)^{n_1}}$$

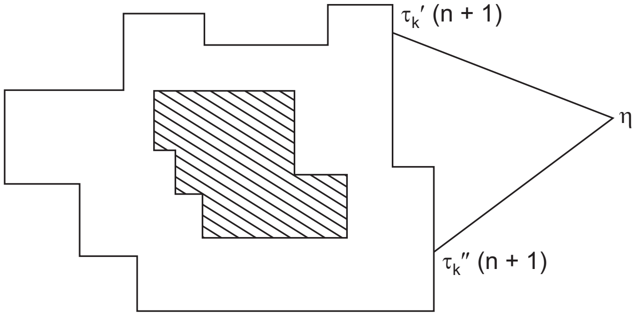


Fig. 14.III

whose only pole is η, such that

$$|\, Q_n(z) - S^{(n+1)}(z)\,| < \varepsilon_n.$$

Here the degree of the polynomial $P_1(z)$ does not exceed n_1.

Hence,

$$|\, f(z) - Q_n(z)\,| < 2\,\varepsilon_n \text{ for all } z \in \overline{\Omega}_n.$$

Observe that $Q_n(z)$ is analytic in $|\, z\,| < \rho_n$ containing $\overline{\Omega}_n$. Hence $Q_n(z)$ can be replaced by power series on $\overline{\Omega}_n$. Let $P_n(z)$ be the sum of a sufficiently large number of initial terms of this series. Thus we find that

$$|\, f(z) - P_n(z)\,| < 3\,\varepsilon_n,\ z \in \overline{\Omega}_n.$$

Hence, the theorem is proved.

Remar: Note that uniform approximation in the complex plane by polynomials has one feature that distinguishes it from the real line. If a sequence of analytic functions converges uniformly inside a region Ω, the limit of the sequence is analytic in Ω. Thus, in Ω, at least, only analytic functions can be approximated uniformly by polynomials. However, this does not rule out the possibility of more general functions being approximated uniformly on sets of different kinds which were not discussed in this chapter. A half century of work in this direction by mathematicians like **Runge, Walsh, Levrentieff, Keldysch** and **Mergelyan** is worth reading. The interested reader may see Davis [8] and Markushevich [11] and some of the references given there.

EXERCISES

1. Consider the weight factors φ_k that appear in the nth Bernstein polynomials. Show that φ_k takes its supremum on $I = [0, 1]$ at the point k/n. Write out explicitly the functions φ_k, $k = 0, 1, 2$, when $n = 2$ and the functions corresponding to $n = 3$, and note that $\Sigma\, \varphi_n(x) = 1$, for $x \in I$. Draw graphs of some of these functions.

2. Let $f(x) \in C^1[a, b]$. If $p(x)$ is a polynomial that approximates f' to within ε on $[a, b]$ then

$$q(x) = \left(\int_a^{\alpha} p(x)\, dx \right) + f(a)$$

is a polynomial that approximates f to within $(b - a)\, \varepsilon$ on $[a, b]$. Extend to higher derivatives.

3. Let $f(x) \in C^\infty [a, b]$. Show (without using Bernstein polynomials) that we can find a sequence of polynomials $p_n(x)$ such that

$$\lim_{n \to \infty} p_n^{(j)}(x) = f^{(j)}(x)$$

uniformly on $[a, b]$, $j = 0, 1, \ldots$.

4. Let

$$p_{nk}(x) = \binom{n}{k} x^k (1 - x)^{n-k},\ 0 \le x \le 1$$

Prove that the maximum value occurs at $x = k/n$. In $k/n \to x$ as $n \to \infty$, $p_{nk}(k/n)$ is asymptotically equal to $(2n\, \pi\, x\, (1 - x))^{-1}$.

5. Bernstein polynomials over the interval $[a, b]$ may be defined by

$$B_n(f, a, b, x) = \frac{1}{h^n} \sum_{k=0}^{n} f\left(a + \frac{k}{n} h \right) \binom{n}{k} (x - a)^k (b - x)^{n-k},$$

$h = (b - a)$. Let $f(x)$ be bounded on $[a, b]$. Prove that

$$\lim_{n \to \infty} B_n(f, a, b, x) = f(x).$$

6. Compare

$$B_4\left(\sqrt{x}, \frac{1}{2} \right) \text{ with } \sqrt{\frac{1}{2}}.$$

7. Let $f(x) \in \mathbb{C}[a, b]$. Prove that $f(x)$ is uniformly approximable on $[a, b]$ by polynomials with rational coefficients.

8. Let $f(x) \in \mathbb{C}[a, b]$. Prove that uniform approximation by polynomials with integer coefficients is not necessarily possible.

APPENDIX

1

RIEMANN MAPPING THEOREM

In order to prove Riemann Mapping theorem we need some important results like open mapping theorem, **Schwarz Lemma** and a basic theorem on **Normal Families.**

We shall first prove a lemma.

Lemma 1 : *Let $D(z_0, \rho)$ be an open disk centred at z_0 and radius ρ. Suppose f is analytic in $D(z_0, \rho)$. Let $f(z_0) = \beta$. If $f(z) - \beta$ has a zero of order m at $z = z_0$ then there is an $\varepsilon > 0$ and $\delta > 0$ such that for $|\xi - \beta| < \delta$, $f(z) = \xi$ has exactly m simple roots in $D(z_0, \varepsilon)$.*

Proof : Since the zeros of an analytic function are isolated, $f(z) = \beta$ has no solutions for $0 < |z - z_0| < 2\varepsilon$, where $\varepsilon > 0$ and $\varepsilon < \frac{1}{2}\rho$. Also, $f'(z) \neq 0$ for $0 < |z - z_0| < 2\varepsilon$. Let $\gamma(t) = z_0 + \varepsilon \exp(2\pi i t)$, $0 \le t \le 1$, and let $\sigma = f \circ \gamma$. Now $\beta \notin \{\sigma\}$, thus there is a $\delta > 0$ such that

$$D(\beta, \delta) \cap \{\sigma\} = \phi.$$

Hence,

$$|\beta - \xi| < \xi$$

implies $n(\sigma ; \beta) = n(\sigma ; \xi) = \sum_{k=1}^{p} n(\gamma ; z_k(\xi))$. This is due to the fact that $D(\beta, \delta) \subset$ in the same component of $C - \{\sigma\}$. Since $n(\gamma ; z)$ is either zero or one, we have $f(z) = \xi$ has m roots in $D(z_0, \varepsilon)$. Each of these roots $(\xi \neq \beta)$ are simple because $f'(z) \neq 0$ for $0 < |z - z_0| < \varepsilon$.

Note that this lemma says that

$$D(\beta, \delta) \subset f(D(z_0, \varepsilon)).$$

Also, since $f(z) - \beta$ have a zero of finite multiplicity, it follows that f is not constant.

Open Mapping Theorem: *Let Ω be a region and let f be non-constant function which is analytic in Ω. Then f(U) is open for any open set U in Ω.*

Proof : Let $U \subset \Omega$ be open. The theorem will be proved if we can show that for each z_0 in U there is a $\delta > 0$ such that $D(\beta, \delta) \subset f(U)$, where $\beta = f(z_0)$. It follows from the Lemma 1 that we need to find an $\varepsilon > 0$ and a $\delta > 0$ such that

$$D(z_0, \varepsilon) \subset U$$

and

$$D(\beta, \delta) \subset f(D(z_0, \varepsilon)).$$

Let X_1 and X_2 be metric spaces. Let $f : X_1 \to X_2$. Suppose that $f(U)$ is open in X_2 whenever U is open in X_1, then f is called an open map. Let f be one-one and onto. We define the inverse map $f^{-1} : X_2 \to X_1$ by

$$f^{-1}(w) = z \text{ where } f(z) = w.$$

It follows that f^{-1} is continuous when f is open. In other words, for $U \subset X_1$, $(f^{-1})^{-1}(U) = f(U)$.

Corollary: *Let Ω be a region. Let $f : \Omega \to C$ be one-one and analytic. Suppose that $f(\Omega) = X_2$. Then $f^{-1} : X_2 \to C$ is analytic and $(f^{-1})'(w) = [f'(z)]^{-1}$ where $f(z) = w$.*

Schwarz Lemma: *Let $D(0,1)$ be an open disk centred at 0 and radius 1. Let f be analytic in $D(0, 1)$. Suppose that*

(*i*) $|f(z)| \leq 1$ *for* $z \in D(0, 1)$

(*ii*) $f(0) = 0$.

Then

(*iii*) $|f(z)| \leq |z|$ *for* $z \in D(0, 1)$

(*iv*) $|f'(0)| \leq 1$;

if equality holds in (*iii*) *for one* $z \in D(0, 1) \sim \{0\}$, *or if equality holds in* (*iv*), *then*

$$f(z) = \lambda z, \text{ where } \lambda \text{ is a constant, } |\lambda| = 1.$$

Proof : Define $g : D(0, 1) \to C$ by

$$g(z) = f(z)/z \text{ for } z \neq 0$$

and

$$g(0) = f'(0);$$

then g is analytic in $D(0, 1)$. By **Maximum Modulus Theorem** $|g(z)| \leq r^{-1}$ for $|z| \leq r$ and $0 < r < 1$. As $r \to 1$, we have $|g(z)| \leq 1$ for all $z \in D(0, 1)$. In other words, $|f(z)| \leq |z|$ and $|f'(0)| = |g(0)| \leq 1$.

If $|f(z)| = |z|$ for some $z \in D(0,1)$, $z \neq 0$, or $|f'(0)| = 1$. Then g assumes its maximum value inside D. Thus, again by Maximum Modulus theorem, $g(z) = \lambda$ for some constant λ with $|\lambda| = 1$. Hence,

$$f(z) = \lambda z \text{ and the proof is complete.}$$

We need Schwarz's Lemma to characterize the **conformal maps** of the open unit disk $D(0, 1)$ onto itself. Let us introduce a class of such maps. If $|\alpha| < 1$, we define the Möbius transformation

$$(1)\ q_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha} z}.$$

Note that q_α maps $D(0,1)$ onto itself and is one-to-one.

Let θ be a real number. Then

$$|q_\alpha(e^{i\theta})| = \left|\frac{e^{i\theta}-\alpha}{1-\overline{\alpha}e^{i\theta}}\right| = \left|\frac{e^{i\theta}-\alpha}{e^{-i\theta}-\overline{\alpha}}\right| = 1.$$

We summarize the above discussion in Lemma 2.

Lemma 2: *If* $|\alpha| < 1$, *define* q_α *as in* (*1*). *Then* q_α *is one-to-one map of D(0, 1) onto itself. The inverse of* q_α *is* $q_{-\alpha}$. *Furthermore, we have*

$$q'_\alpha(0) = 1-|\alpha|^2,$$

and

$$q'_\alpha(\alpha) = (1-|\alpha|^2)^{-1}.$$

Since q_α is analytic for $|z| < \dfrac{1}{|\alpha|}$, it is analytic in an open disk $D_1 \supset \overline{D}(0, 1)$.

Also, it can be easily seen that

$$q_\alpha(q_{-\alpha}(z)) = z = q_{-\alpha}(q_\alpha(z)) \text{ for } |z| < 1.$$

We also find that

$$q'_\alpha(0) = 1-|\alpha|^2$$

and

$$q'_\alpha(\alpha) = \frac{1}{1-|\alpha|^2}.$$

Definition (*Normal Families*) : Let Ω be a region. Suppose $\mathcal{F} \subset H(\Omega)$. If every sequence of numbers of $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of Ω, then we call $\mathcal{F}$ a normal family.

Note that in the definition the limit function is not required to belong to $\mathcal{F}$.

Theorem 1. *Let* $\mathcal{F} \subset H(\Omega)$. *Suppose that* $\mathcal{F}$ *is uniformly bounded on each compact subset of* Ω. *Then* $\mathcal{F}$ *is a normal family.*

Proof: By hypothesis we have to each compact set $E \subset \Omega$ there corresponds a number $K(E) < \infty$ such that

$$|f(z)| \le K(E)$$

for all $f \in \mathcal{F}$ and all $z \in E$.

Let $\{E_n\}$ be a sequence of compact sets whose union is Ω such that $E_n \subset$ int. E_{n+1}. Then there exists positive numbers δ_n such that

(2) $D(z, 2\delta_n) \subset E_{n+1}$, $z \in E_n$.

Consider two points z' and z'' in E_n where $|z'-z''| < \delta_n$. Since

$$\frac{1}{\xi-z'} - \frac{1}{\xi-z''} = \frac{z'-z''}{(\xi-z')(\xi-z'')},$$

by using Cauchy formula we have

$$(3) \quad f(z')-f(z'') = \frac{z'-z''}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi-z')(\xi-z'')}\,d\xi$$

where γ is a circle with centre at z' and radius $2\delta_n$. Since $|\xi - z'| = 2\delta_n$ and $|\xi - z''| > \delta_n$ for all $\xi \in \gamma(t)$, (3) gives

$$(4) \quad |f(z') - f(z'')| < \frac{K(E_{n+1})}{\delta_n} |z' - z''|.$$

The inequality (4) is valid for all $f \varepsilon \mathcal{F}$ and z' and $z'' \in E_n$, provided that $|z' - z''| < \delta_n$.

By taking

$$(5) \quad \delta = \frac{\varepsilon \cdot \delta_n}{K(E_{n+1})}$$

in (4) we get to each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that $|f(z') - f(z'')| < \varepsilon$ for all $f \in \mathcal{F}$ and all z' and $z'' \in E_n$ for which $|z' - z''| < \delta$.

Now let $\{f_m\}$ be a sequence in $\mathcal{F}$. Let us choose a countable dense subset $\{z_1\}$ of Ω. By hypothesis $\{f_m(z)\}$ is bounded at each $z \in \Omega$. Hence $\{f_m\}$ has a subsequence which converges. Denote this subsequence as $\{f_{m,1}\}$ which converges at z_1. From $\{f_{m,1}\}$ we can extract a subsequence denoted by $\{f_{m,2}\}$ which also converges at z_2. Proceeding in this manner we obtain sequences $\{f_{m,i}\}$ which converges at z_i where $\{f_{m,i}\}$ is a subsequence of $\{f_{m,i-1}\}$. Thus the sequence $\{f_{m,n}\}$ which is called the **diagonal sequence** converges at every one of the points z_i.

Now we have to prove that $\{f_{m,n}\}$ converges uniformly on each E_n.

We fix E_n and also fix $\varepsilon > 0$. Choose δ as in (5). Then there are points $z_1, z_2, \ldots, z_p$ of the set $\{z_i\}$ such that

$$E_n \subset \bigcup_{i=1}^{p} \{D(z_i, \delta)\}$$

and there is an integer N such that

$$(6) \; |f_{\mu,\mu}(z_i) - f_{\nu,\nu}(z_i)| < \varepsilon$$

where $\mu > N$, $\nu > N$, and $1 \le i \le p$.

We have for every $z \in E_n$ there corresponds a z_i such that $1 \le i \le p$ and $|z - z_i| < \delta$.

Then

$$(7) \; |f_{\mu,\mu}(z) - f_{\nu,\nu}(z)|$$
$$\le |f_{\mu,\mu}(z) - f_{\nu,\nu}(z_i)| + |f_{\mu,\mu}(z_i) - f_{\nu,\nu}(z_i)| + |f_{\mu,\mu}(z_i) - f_{\nu,\nu}(z)|.$$

If $\mu > N$ and $\nu > N$, the second term in the right of (7) is less than ε. The first and third term in the right of (7) is less than ε by our choice of δ. Hence

$$(8) \; |f_{\mu,\mu}(z) - f_{\nu,\nu}(z)| < 3\varepsilon$$

for all $z \in E_n$, if $\mu > N$ and $\nu > N$.

Thus $\{f_{m,n}\}$ converges uniformly on each E_n and hence on each compact subset E of Ω. The proof of the theorem is now complete.

Definition (Conformal Equivalence) : Let Ω_1 and Ω_2 be two regions. We say that Ω_1 is **conformally equivalent** to Ω_2 if there is a function $f \in H(\Omega_1)$ such that f is one-one and $f(\Omega_1) = \Omega_2$.

Note that this is an equivalence relation. It follows from Liouville's theorem that C is not equivalent to any bounded region. It also follows from the definitions that if Ω_1 is simply connected and Ω_1 is equivalent to Ω_2 then Ω_2 is simply connected.

Theorem 2 (Riemann Mapping). *Every simply connected region Ω in C ($\Omega \neq C$) is conformally equivalent to the open unit disk $D(0, 1)$.*

Before carrying out the proof of this important theorem we would like to point out that the only property of a simply connected region which will be used is that every analytic function which has no zero in such a region has an analytic square root there.

Proof : Suppose Ω is a simply connected region in C. Let w_0 be a complex number and $w_0 \notin \Omega$. Let Φ be the class of all $\psi \in H(\Omega)$ such that ψ is one-one and map Ω into $D(0, 1)$. We need to prove that some $\psi \in \Phi$ maps Ω onto $D(0, 1)$.

We first prove that Φ is non-empty. Since Ω is simply connected there is a function $f \in H(\Omega)$ such that

$$f^2(z) = z - w_0 \text{ in } \Omega.$$

If $f(z_1) = f(z_2)$, then $f^2(z_1) = f^2(z_2)$. Hence $z_1 = z_2$ and thus f is one-one. By similar argument we find that

$$f(z_1) = -f(z_2), \text{ where } z_1, z_2 \notin \Omega$$

Since f is an open mapping, $f(\Omega) \supset D(a, r)$ where $0 < r < |a|$. The disk $D(-a, r)$ therefore does not intersect $f(\Omega)$. By putting

$$\psi = r/(f + a)$$

we see that $\psi \in \Phi$.

We now prove that if $\psi \in \Phi$, if $\psi(\Omega)$ does not cover all of $D(0, 1)$, and if $z_0 \in \Omega$, then there is a $\psi_1 \in \Phi$ with

$$|\psi'_1(z_0)| > |\psi'(z_0)|.$$

Define q_α as defined in (1) *i.e.*

$$q_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha}z}.$$

It follows from Lemma 2 that for $\alpha \in D(0, 1)$, q_α is one-one mapping of $D(0,1)$ onto $D(0, 1)$, its inverse mapping is $q_{-\alpha}$. Suppose $\psi \in \Phi$, $\alpha \in D(0, 1)$ and $\alpha \notin \psi(\Omega)$. Then $q_\alpha \circ \psi \in \Phi$, and $q_\alpha \circ \psi$ has no zero in Ω. Thus there is a function $g \in H(\Omega)$ such that $g^2 = q_\alpha \circ \psi$. We find that g is one-one and since Φ is non-empty we have $g \in \Phi$.

Let $\psi_1 = g \circ q_\beta$ where $\beta = g(z_0)$. It follows that $\psi_1 \in \Phi$. Denote $w^2 = s(w)$. We now have

$$\psi = q_{\alpha} \circ s \circ g = q_{-\alpha} \circ s \circ q_{-\beta} \circ \psi_1.$$

Since $\psi_1(z_0) = 0$, by Chain rule we get

$$\psi'(z_0) = F'(0)\, \psi'_1(z_0), \text{ where}$$

$$F = q_{-\alpha} \circ s \circ q_{-\beta}.$$

Hence $F\{D(0, 1)\} \subset D(0, 1)$ and F is not one-one in $D(0, 1)$. Therefore, by using Schwarz Lemma, we get

$$| F'(0) | < 1.$$

Hence $$| \psi'(z_0) | < | \psi'_1(z_0) |.$$

We now fix $z_0 \in \Omega$, and put

$$\eta = \sup \{| \psi'(z_0) | : \psi \in \Phi\}.$$

We saw in the preceding discussion that any function $h \in \Phi$ for which $| h'(z_0) | = \eta$ will map Ω onto $D(0, 1)$. The proof of the theorem will be complete if we prove the existence of such a function h.

Since $| \psi(z) | < 1$ for all $\psi \in \Phi$ and $z \in \Omega$. It follows from Theorem 1 that Φ is a normal family. By the definition of η it follows that there is a sequence $\{\psi_n\}$ in Φ such that $| \psi'_n(z_0) | \to \eta$, and since Φ is normal. We can extract a subsequence $\{P_n\}$ from $\{\psi_n\}$ which converges uniformly on compact subsets of Ω to a limit function $h \in H(\Omega)$. By theorem 5 of Chap. 10, $| h'(z_0) | = \eta$. Since Φ is non-empty, $\eta > 0$, so h is not constant.

Since $P_n(\Omega) \subset D(0,1)$, we have $h(\Omega) \subset \overline{D}(0, 1)$. Hence by the open mapping Theorem we get $h(\Omega) \subset D(0, 1)$. Now it remains to prove that h is one-one.

Let z_1 and z_2 be fixed points belonging to Ω. Put $\alpha = h(z_1)$ and $\alpha_n = P_n(z_1)$. Let $\overline{D}(z_2, r)$ be a closed circular disk contained in Ω such that $z_1 \notin \overline{D}(z_2, r)$. Suppose also that $h - \alpha$ has no zero on the boundary of $\overline{D}(z_2, r)$. Since the zeros of $h - \alpha$ have no limit point in Ω, the above supposition makes sense. The function $\psi_n - \alpha_n$ converges to $h - \alpha$ uniformly in $\overline{D}(z_2, r)$, they have no zero in $D(0, 1)$. Since they are one-one and have a zero at z_1, it follows from Rouche's Theorem that $h - \alpha$ has no zero in $D(0, 1)$. Thus, in particular, we have $h(z_1) \neq h(z_2)$.

Hence $h \in \Phi$, and the proof of the Riemann mapping theorem is complete.

APPENDIX

2

HOMOLOGICAL VERSION OF CAUCHY'S THEOREM

We recall some definitions. Let Ω be an open set. Let γ, τ be closed paths in Ω, γ and τ are said to be homologous in Ω if

$$n(\gamma, \alpha) = n(\tau, \alpha)$$

for every $\alpha \notin \Omega$.

A closed path γ in Ω is homologous to 0 in Ω if

$$n(\gamma, \alpha) = 0 \text{ for every } \alpha \notin \Omega.$$

Let $\gamma_1, \gamma_2, ..., \gamma_n$ be curves, and let $m_1, m_2, ..., m_n$ be integers. A sum of the form

$$\gamma = \sum_{i=1}^{n} m_i \gamma_i$$

is called a chain. We say that γ is a chain in Ω if each γ_i is in Ω. A chain is said to be closed if it is a finite sum of closed paths.

We define

$$\int_\gamma f = \sum_{i=1}^{n} m_i \int_\gamma f$$

where γ is a chain.

If γ is a closed chain where each γ_i is a closed path, then the index of γ with respect to a point α is defined as

$$n(\gamma, \alpha) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - \alpha}$$

where α is not on the chain.

We write

$$\gamma \sim \tau$$

if $n(\gamma, \alpha) = n(\tau, \alpha)$ for every $\alpha \notin \Omega$ and

$$\gamma \sim 0$$

if $n(\gamma, \alpha) = 0$ for every $\alpha \notin \Omega$.

Let Ω be an open set. Let γ, τ be two paths in Ω. Suppose that γ, τ are defined on the same interval [0, 1]. We say that γ, τ are "near-together" if there exists a partition

$$0 = u_0 \leq u_1 \leq u_2 \leq \ldots \leq u_n = 1$$

and for each $j = 0, 1, 2, \ldots, n - 1$, there exists a disc D_j contained in Ω such that

$$\gamma(\{u_j, u_{j+1}]) \subset D_j$$

and
$$\tau([u_j, u_{j+1})] \subset D_j.$$

Let Ω be an open set. Let γ be a curve in Ω, defined on [0, 1]. Let

$$0 = u_0 \leq u_1 \leq u_2 \leq \ldots \leq u_n = 1$$

be a partition of [0, 1]. Let

$$\gamma_j : [u_{j,} u_{j+1}] \to \Omega$$

be the restriction of γ to the smaller interval $[u_j, u_{j+1}]$. Then the chain

$$\gamma_1 + \gamma_2 + \ldots + \gamma_n$$

is called a subdivision of γ.

By another parameterization, if τ_j is obtained from γ_j, then the chain

$$\tau_1 + \tau_2 + \ldots + \tau_n$$

is also a subdivision of γ.

Note that the chain γ and

$$\tau_1 + \tau_2 + \ldots + \tau_n$$

do not differ from each other. We illustrate this in Fig. II.I

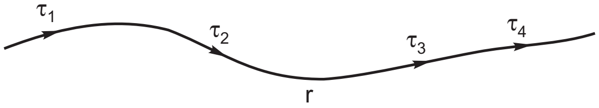


Fig. Appendix II.I

Suppose that $\{\tau_{ji}\}$ is a subdivision of γ_j where

$$\gamma = \sum m_j \gamma_j.$$

Then
$$\sum_j \sum_i m_j \tau_{ji}$$

is called a subdivision of γ.

We say that a path is rectangular if every curve of the path is either a horizontal line segment or a vertical line segment.

In the following lemma we shall show that every path is homologous with a rectangular path.

Lemma: *Let Ω be an open set. Let γ be a closed path in Ω. Then there exists a rectangular closed path τ with the same terminal points, and such that γ, τ are "near together" in Ω.*

In particular, $\gamma \sim \tau$ in Ω, and if f is analytic on Ω, then

$$\int_\gamma f = \int_\tau f.$$

Proof : Let γ be defined on an interval [0, 1]. Choose a partition

$$0 = u_0 \leq u_1 \leq \ldots \leq u_n = 1$$

of [0, 1] such that the image of each small interval ($[u_j, u_{j+1}]$) is contained in a disc D_j on which f has a primitive. Replace the curve γ on the interval $[u_j, u_{j+1}]$ by the rectangular curve as shown in Fig. II.II.

We now set $z_j = \gamma(u_j)$ in Fig. II.II.

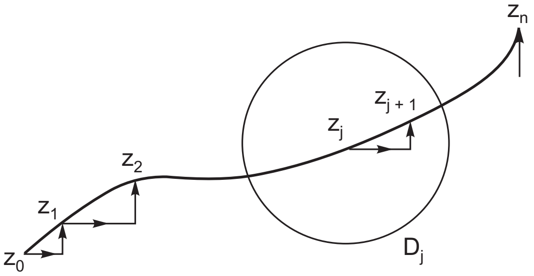


Fig. Appendix II.II

Note that if γ is a closed path, then the rectangular path constructed in the lemma is also a closed path. The closed rectangular path is illustrated in Fig. II.III.

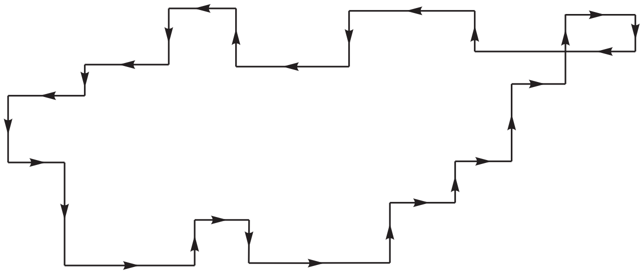


Fig. Appendix II. III

We now prove a very important theorem which has nothing to do with analytic functions. The proof of this remarkable theorem is due to **Artin.** We shall see that **Homological version** of Cauchy's theorem follows immediately from this theorem.

Theorem 1. *Let Ω be an open set. Let γ be a rectangular closed chain in Ω. Assume that $\gamma \sim 0$ in Ω. Then there exists rectangles $T_1, T_2, \ldots, T_n$* contained *in Ω such that a subdivision of γ is equal to $\sum_{j=1}^{N} m_j \, \partial T_j$ where ∂T_j is the boundary of T_j* (*oriented counterclockwise*) *and m_j are some integers.*

Proof : Let the rectangular chain γ be given. We draw lines parallel to both axes passing through the sides of the chain (Fig. II.IV). Then these vertical and horizontal lines decompose the plane into finite rectangles and some unbounded regions which may be considered as rectangular regions extending to infinity in the vertical and horizontal direction. Let T_j be one of the finite rectangles. Choose a point α_j from the interior of T_j and let

$$m_j = n(\gamma, \alpha_j).$$

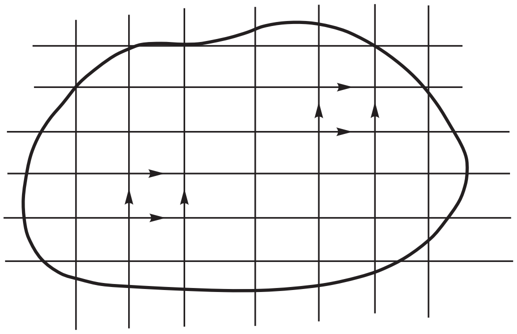


Fig. Appendix II.IV

It follows that for some rectangles $m_j = 0$, and for some rectangles $m_j \neq 0$. Let $m_j \neq 0$. $j = 1, 2, 3, ..., N$ for rectangles $T_j, j = 1, 2, ..., N$ and let ∂T_j be the corresponding boundary (oriented counterclockwise). We shall first show that every rectangle T_j such that $m_j \neq 0$ is contained in Ω.

By our hypothesis α_j must lie in Ω, because $n(\gamma, \alpha) = 0$ for $\alpha \notin \Omega$. Since Ω is connected, the index (winding number) is constant on Ω, and hence it is constant on the interior of T_j. This shows that the index (winding number) is not zero and the interior of T_j is contained in Ω. It is obvious that if a boundary point of T_j is on γ, then it is in Ω. If a boundary point of T_j is not on γ, then it follows from Lemma 2 of Chapter 5 that $m_j \neq 0$. Thus the rectangle T_j with its boundary is contained in Ω.

We now prove that some subdivision of γ is equal to

$$\sum_{j=1}^{N} m_j \, \partial T_j .$$

In order to prove this we need to find first an appropriate subdivision of γ.

Since the vertical and horizontal line cut γ at several points, we can find a subdivision τ of γ such that every curve in τ is some side of a finite rectangle, or the finite side of the infinite rectangles. The subdivision τ is the sum of such sides as described above.

It is now sufficient to prove that

$$\tau = \sum m_j \partial T_j.$$

Let γ_0 be the closed chain defined by

$$\gamma_0 = \tau - \sum_{j=1}^{N} m_j \, \partial T_j .$$

We claim that $n(\gamma_0, \alpha) = 0$ for every $\alpha \notin \gamma_0$.

We consider various possible cases to prove our claim.

If α lies in one of the infinite rectangles, then it follows from Lemma 3 of Chapter 5 that $n(\gamma_0, \alpha) = 0$. Suppose α lies in some infinite rectangle T. If T is one of the rectangles T_j, say $T = T_k$, then

$$n(\partial T_k, \alpha) = 1$$

and $$n(\partial T_k, \alpha) = 0 \text{ if } j \neq k.$$

Applying Lemma 3 of Chapter 5 again and using the definition of m_j we get

$$n(\gamma_0, \alpha) = n(\gamma, \alpha) - m_k n(\partial T_k, \alpha) = 0.$$

Suppose that T is one of the rectangles T_j, it follows from the definition that

$$n(\gamma, \alpha) = 0,$$

and by Lemma 3 of Chapter 5 that

$$n(T_j, \alpha) = 0 \text{ for all } j = 1, 2, ..., N.$$

Thus in this case also we have

$$n(\gamma, \alpha) = 0.$$

Suppose α lies on the boundary of a finite rectangle or an infinite rectangle, but not on γ_0. By the continuity of the index (winding number) we find that $n(\gamma_0, \alpha) = 0$. We thus proved that if γ_0 is a closed chain defined by

$$\gamma_0 = \tau - \sum_{j=1}^{N} m_j \, \partial T_j$$

then $$n(\gamma_0, \alpha) = 0 \text{ for } \alpha \notin \gamma_0.$$

It remains to prove that $\gamma_0 = 0$.

Suppose that $\gamma_0 \neq 0$. Write

$$\gamma_0 = m\sigma + \gamma_0$$

where σ is a horizontal or vertical segment, γ_0 is a chain of vertical and horizontal segments other than σ and m is a non-zero integer. We have to consider two cases. In the first case σ is the side of some finite rectangle and in the second case σ is the side of two infinite rectangular regions. We illustrate these two cases in the following Fig. II.V.

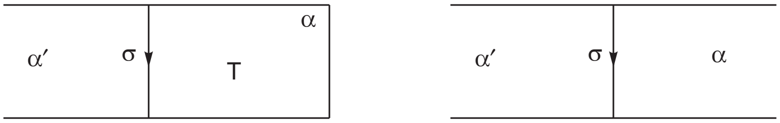


Fig. Appendix II.V

In the first case, the chain $\gamma_0 - m\,\partial\,T$ does not contain σ. Suppose that α is a point inside T and α' is a point on the opposite side from α but near to σ. Then there is a line segment which joins α and α' but does not intersect $\gamma_0 - m\,\partial\,T$. It now follows from the connectedness and the continuity of the line segment that

$$n(\gamma_0 - m \cdot \partial T, \alpha) = n(\gamma_0 - m \cdot \partial T, \alpha').$$

Note that by definition,

$$n(m \cdot \partial T, \alpha) = m,$$

and

$n(m \cdot \partial T, \alpha') = 0$, because α' is outside T. Since $n(\gamma_0, \alpha) = n(\gamma_0, \alpha') = 0$, it follows that $m = \theta$.

In the second case, the two regions adjacent to σ are both infinite. We shall see that this is not possible.

By Lemma 3 of Chapter 5

$$n(\gamma_0, \alpha) = n(\gamma_0, \alpha') = 0.$$

Hence,

$$m\int_\sigma \frac{dz}{z-\alpha} + \int_{\gamma_o} \frac{dz}{z-\alpha} = m\int_\sigma \frac{dz}{z-\alpha'} + \int_{\gamma_o} \frac{dz}{z-\alpha'}$$

i.e.

$$(1) \qquad m\int_\sigma \left(\frac{1}{z-\alpha} - \frac{1}{z-\alpha'}\right) dz = \int_{\gamma_o} \left(\frac{1}{z-\alpha'} - \frac{1}{z-\alpha}\right) dz\,.$$

It can be easily seen that the left side of (1) is equal to a fixed non-zero multiple of $2\pi i$. The right side of (1) is a continuous function of α, α' where α, α' approach each other on a line segment passing through σ. This can happen only when $m = 0$.

We thus proved that $\gamma_0 = 0$ and therefore a subdivision of γ is equal to

$$\sum m_j \partial T_{j.}.$$

This proves the theorem.

Cauchy's Theorem (Homological Version)

Let Ω be an open set. Let γ be a closed chain in Ω and let $\gamma \sim 0$ in Ω. Assume that f is analytic on Ω. Then

$$\int_\gamma f = 0.$$

Proof: By the Lemma, it is sufficient to prove the theorem to the case when γ is a rectangular closed chain. Applying Theorem 1 we see that Cauchy's theorem follows easily.

We know that if f is analytic on Ω, then

$$\int_{\partial T_j} f = 0$$

Thus $$\int_{\sum_{j=1}^{N} m_j \partial T_j} f = 0.$$

This prove that $$\int_\gamma f = 0.$$

SOLUTIONS TO SOME SELECTED EXERCISES

1. Let (X, d) be a metric space. Prove that $(X, \frac{d}{1+d})$ is also a metric space.

Solution:

(*i*) $\frac{d(x, y)}{1 + d(x, y)} = 0$ iff $d(x, y) = 0$ iff $x = y$

(*ii*) $\frac{d(x, y)}{1 + d(x, y)} = \frac{d(y, x)}{1 + d(y, x)}$ since $d(x, y) = d(y, x)$

The triangle inequality

(*iii*) $\frac{d(x, z)}{1 + d(x, z)} \le \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)}$

can be verified as follows:

Multiplying both sides of (*iii*) by

$$(1 + d(x, y))\,(1 + d(y, z))\,(1 + d(z, x)) \text{ we obtain}$$

$$d(x, z) \le d(x, y) + d(y, z) + 2d(x, y)\, d(y, z) + d(x, y)\, d(y, z)\, d(z, x)$$

Since d is a distance and is ≥ 0 (*iii*) follows.

2. Let E and F be two compact subsets of the metric space (X, d). Define the distance dist. (E, F) by dist. $(E, F) = \inf \{d(x, y) : x \in E, y \in F\}$. Show that if $E \cap F = \phi$, then dist. $(E, F) > 0$.

Solution: Clearly, dist. $(E, F) \ge 0$. We will show that if $E \cap F = \phi$, then dist. $(E, F) \ne 0$. The proof is by contradiction. Suppose that dist. $(E, F) = 0$. Then for a given n there exists $x_n \in E$ and $y_n \in F$ such that $d(x_n, y_n) \le \frac{1}{n}$. Taking $n = 1, 2, \ldots$ we get a sequence $\{x_n\}$ and a sequence $\{y_n\}$. Since E is compact and $x_n \in E$, we can find a subsequence of $\{x_n\}$ converging $x_0 \in E$. Similarly, there exists a subsequence of $\{y_n\}$ converging to $y_0 \in F$.

Now $\qquad d(x_0, y_0) = \lim\limits_{n \to \infty} (x_n, y_n) = 0$, so $x_0 = y_0$ and $E \cap F \ne \phi$.

3. Prove that the mapping $x \to \frac{1}{x}$, $x \in (0, 1)$ is continuous but not uniformly continuous.

Solution: $\frac{1}{x}$ is continuous for $x \neq 0$ in (0, 1).

Let $\varepsilon > 0$ be given. Then, however small we might choose $\delta(\varepsilon)$, there exists $x_1, x_2 \in (0, 1)$ with $|x_1 - x_2| < \delta(\varepsilon)$ such that

$$\left| \frac{1}{x_1} - \frac{1}{x_2} \right| > \varepsilon$$

Taking $x_2 = x_1 + \frac{1}{2}\delta(\varepsilon)$, we have

$$|x_2 - x_1| < \delta(\varepsilon) \text{ and}$$

$$\left| \frac{1}{x_1} - \frac{1}{x_2} \right| = \frac{1}{2}\delta(\varepsilon)/x_1\left(x_1 + \frac{1}{2}\delta(\varepsilon)\right).$$

But
$$\lim_{x_1 \to 0} x_1 \frac{\frac{1}{2}\delta(\varepsilon)}{\left(x_1 + \frac{1}{2}\delta(\varepsilon)\right)} = \infty.$$

Hence, it is possible to choose $x_1 \in (0, 1)$ such that

$$\left\{ \frac{1}{2}\delta(\varepsilon) / x_1\left(x_1 + \frac{1}{2}\delta(\varepsilon)\right) \right\} > \varepsilon$$

4. Show that $f(z) = |z|$ is nowhere differentiable.

Solution: Let $w = f(z) = u + io = \sqrt{(x^2 + y^2)} = r$.

Now
$$\frac{\partial w}{\partial r} = \frac{\partial r}{\partial r} = 1.$$

This shows that the rate of change in the radial direction is 1. But the derivative in the direction of the tangent

$$\frac{\partial f}{\partial r}\frac{\partial r}{\partial \theta} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial \theta} = 0\ \frac{\partial r}{\partial \theta} = 0.$$

Since the derivative in the radial direction is different from the derivative in the direction of the tangent. Hence $f(z) = |z|$ is nowhere differentiable.

5. Show that the function $f(x, y) = x^2 + iy^2$ is uniformly continuous in the open disc $D(0, 1)$ but not uniformly continuous in C.

Solution: Consider

$$\begin{aligned} &|f(x, y) - f(x_1, y_1)| \\ &\quad = |x^2 + iy^2 - x_1^2 - iy_1^2| \\ &\quad = |(x + x_1)(x - x_1) + i(y + y_1)(y - y_1)| \\ &\quad \leq |(x + x_1)(x - x_1)| + |(y + y_1)(y - y_1)| \\ &\quad \leq 2(|x - x_1| + |y - y_1|), \end{aligned}$$

since $|x + x_1| \leq 2$, $(y + y_1) \leq 2$ in $D(0, 1)$.

Hence

$$|f(x, y) - f(x_1, y_1)| \le \varepsilon$$

for $$\sqrt{(x - x_1)^2 + (y - y_1)^2} \le \varepsilon/4.$$

This shows that $f(x, y)$ is uniformly continuous in $D(0, 1)$.

But $f(x, y)$ is not uniformly continuous in C. For, choose

$$x + x_1 = y + y_1 = M > 0$$

Then

$$|f(x, y) - f(x_1, y_1)| = M\,|(x - x_1) + i(y - y_1)|$$

$$= M\sqrt{(x - x_1)^2 + (y - y_1)^2}$$

Thus, however small we choose $\sqrt{(x - x_1)^2 + (y - y_1)^2}$,

$$|f(x, y) - f(x_1, y_1)| > \varepsilon$$

for sufficiently large M.

6. Show that

$$\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x},\ i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) = \frac{\partial}{\partial y}$$

and $$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$$

Solution:

(1) $\dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)$

(2) $\dfrac{\partial}{\partial \bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$

By adding (1) and (2) we obtain

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$$

and by subtracting (2) from (1) we have

$$\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} = -i\frac{\partial}{\partial y}$$

i.e., $$i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) = \frac{\partial}{\partial y}$$

Now $$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right)$$

$$= \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right)\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right) + i^2\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$$

$$= \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \bar{z}^2} + 2\frac{\partial^2}{\partial z\,\partial \bar{z}} - \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \bar{z}^2} + 2\frac{\partial^2}{\partial z\,\partial \bar{z}} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$$

7. Prove that the function $f(z) = u + iv$ where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}$$

$z \neq 0, f(0) = 0$, satisfies $C - R$ equations at the origin, but $f'(0)$ does not exist.

Solution:

$$u(x, y) = \frac{x^3 + y^3}{x^2 + y^2}$$

and $$v(x, y) = \frac{x^3 + y^3}{x^2 + y^2}$$

$$u(0, 0) = 0, v(0, 0) = 0$$

$$u_x = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = 1$$

$$u_y = \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y} = -1$$

Similarly $v_x = 1, v_y = 1$

Hence, $C - R$ equations are satisfied at the origin.

But $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z \to 0} \frac{[x^4 + y^4 + ixy(y^2 - x^2)](1+i)}{(x^2 + y^2)^2}$$

Letting $z \to 0$ along $y = x$, we have

$$f'(0) = \frac{1}{2}(1 + i)$$

But $$f'(0) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)_{(0,0)} = 1 + i$$

Hence $f'(0)$ is not unique. This shows that $f(z)$ is not analytic at the origin but $C - R$ equations are satisfied there.

8. Prove that $f(z) = z\bar{z}$ is nowhere analytic

Solution: $f(z) = z\bar{z} = x^2 + y^2 = u + i0$

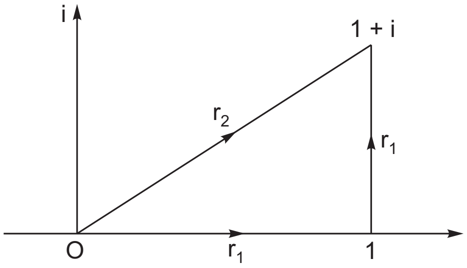
The first four partial derivatives

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 2y, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 0$$

are continuous everywhere. But $C - R$ equations are satisfied only at the origin. Hence $z = 0$ is the only point at which $f'(z)$ exists. Thus $f(z) = z\bar{z}$ is nowhere analytic.

9. Evaluate $\int_0^{1+i} (z^2 + z)\, dz$. By choosing two different paths of integration show that the results are the same.

Solution: Choose two paths γ_1 and γ_2 as shown in the figure.



Now $$\int_{\gamma_1} (z^2 + z)\, dz = \int_0^1 (x^2 + x)\, dx + \int_0^1 [(1 + iy)^2 + (1 + iy)]^2\, idy$$

$$= \frac{1}{3} + \frac{1}{2} + \left(2 - \frac{1}{3}\right)i - \frac{3}{2}$$

$$= \frac{5}{3}i - \frac{2}{3}$$

Let $z = (1 + i)\, t$, $0 \le t \le 1$. Then $dz = (1 + i)dt$ and

$$\int_{\gamma_2} (z^2 + z)\, dz = (1 + i) \int_0^1 [(1 + i)^2 t^2 + (1 + i)t]\, dt$$

$$= \frac{(1 + i)^3}{3} + \frac{(1 + i)^2}{2}$$

$$= \frac{5i}{2} - \frac{2}{3}$$

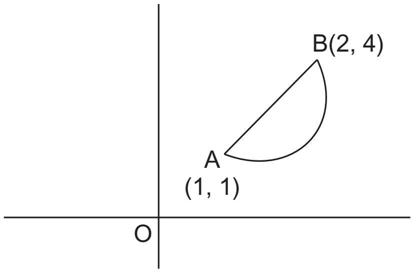
10. Evaluate $\int z^2\, dz$ along

(*i*) the line segment joining the points (1, 1) and (2, 4) (*ii*) the curve $x = t$, $y = t^2$ joining the points (1, 1) and (2, 4).

Solution: (*i*) The equation of the line joining (1, 1) and (2, 4) is $y = 3x - 2$. Any point on this line is given by

$$z = x + i\,(3x - 2),\ 1 \le x \le 2.$$

Hence $$dz = dx + idy = (1 + 3i)\, dx$$



and
$$z^2 = x^2 - y^2 + 2ixy$$
$$= (-8 + 6i)\,x^2 + (12 - 4i)x - 4$$

Thus
$$\int_{AB} f(z)\,dz = \int_1^2 [(-8 + 6i)x^2 + (12 - 4i)x - 4]\,(3 + i)\,dx$$
$$= -\frac{86}{3} - 6i$$

(*ii*) Any point on the arc of the given curve is
$$z = t + it^2,\ 1 \le t \le 2$$
$$\frac{dz}{dt} = 1 + 2it$$

Now integrating $f(z) = z^2$ along the curve, we have
$$\int_\gamma f(z)\,dz = \int_1^2 (t + it^2)^2\,(1 + 2it)\,dt$$
$$= -6i - \frac{86}{3}.$$

This shows that the value of the integral along the two paths joining (1, 1) and (2, 4) is the same.

11. Show that $\left|\int_\gamma (x^2 - iy^2)\,dz\right| \le 2.5$

where

(*i*) γ is the interval $[-i, i]$ on the y-axis

(*ii*) γ is the semi-circle $z = \cos\varphi + i\sin\varphi$, $-\pi/2 \le \varphi \le \pi/2$

Solution:

(*i*) We will use the inequality
$$\left|\int_\gamma f(z)\,dz\right| \le \max_{z\in\gamma} |f(z)|\,L(\gamma)$$

where $L(\gamma)$ is the length of γ.

On $[-i, i]$, $x^2 = 0$ and $|iy^2| \le 1$.

Hence $|x^2 + iy^2| \le 1$. Since $L(\gamma) = 2$, it follows that
$$\left|\int_\gamma (x^2 - iy^2)\,dz\right| \le 1.2 < 2.5$$

(*ii*) In this case $x = \cos\varphi$, $y = \sin\varphi$,
$$dz = (-\sin\varphi + i\cos\varphi)\,d\varphi$$

Now
$$\int_\gamma (x^2 - iy^2)\,dz = \frac{2}{3} + \frac{4}{3}i$$

Hence
$$\left|\int_\gamma (x^2 - iy^2)\,dz\right| = \frac{2}{3}\sqrt{5} < 2.5$$

12. Prove that homotopy with respect to a domain Ω is a transitive property.

Solution: Let $\gamma_1, \gamma_2, \gamma_3$ be three closed curves in a domain Ω. Assume that γ_1 and γ_2 are homotopic with respect to Ω, with the family of curves $\gamma_{12}(t, u)$, $0 \le t \le 1$, and $0 \le u \le 1$ generating the homotopy, so that

$$\gamma_{12}(t, 0) = \gamma_1 \quad \text{and} \quad \gamma_{12}(t, 1) = \gamma_2.$$

Similarly, we assume that γ_2 and γ_3 are homotopic with respect to Ω where $\gamma_{23}(t, 0) = \gamma_2$ and $\gamma_{23}(t, 1) = \gamma_3$.

Now, by choosing properly the parameter t it can be arranged that $\gamma_{12}(t, 1) = \gamma_{23}(t, 0)$.

Define $\gamma_{13}(t, u)$ by

$$\gamma_{13}(t, u) = \begin{cases} \gamma_{12}(t, 2u) & \text{for } 0 \le u \le \dfrac{1}{2} \\ \gamma_{23}(t, 2u - 1) & \text{for } \dfrac{1}{2} \le u \le 1 \end{cases}$$

$\gamma_{13}(t, u)$ is continuous in $0 \le t \le 1$, $0 \le u \le 1$ and

$$\gamma_{13}(t, 0) = \gamma_{12}(t, 0) = \gamma_1,$$

$$\gamma_{13}\left(t, \frac{1}{2}\right) = \gamma_{12}(t, 1) = \gamma_{23}(t, 0) = \gamma_2,$$

$$\gamma_{13}(t, 1) = \gamma_{23}(t, 1) = \gamma_3.$$

Note that they all are within the domain Ω. Hence γ_1 is homotopic to γ_3 with respect to Ω.

13. Evaluate

$$\int_\gamma \frac{e^z}{z^2 + 1}\, dz$$

if γ is the circle of radius 1 at (*i*) $z = i$ and (*ii*) $z = -i$.

Solution:

(*i*)
$$\int_\gamma \frac{e^z}{z^2 + 1}\, dz = \int_\gamma \frac{e^z}{z + i} \frac{dz}{z - i}$$

Identify z_0 as i and $f(z)$ as $\dfrac{e^z}{z + i}$. Applying Cauchy's integral formula we obtain

$$\int_\gamma \frac{e^z}{z + i} \frac{dz}{z - i} = 2\pi\, i f(z_0) = 2\pi i \frac{e^i}{2i}$$
$$= \pi(\cos 1 + i \sin 1)$$

(*ii*) Identify z_0 as $-i$ and $f(z)$ as $\dfrac{e^z}{z - 1}$.

Applying Cauchy's integral formula we obtain

$$\int_\gamma \frac{e^z}{z - i} \frac{dz}{z + i} = 2\pi\, i f(z_0) = 2\pi i \frac{e^{-i}}{-2i}$$
$$= -\pi(\cos 1 - i \sin 1)$$

14. Prove that Liouville's theorem remains true if we replace the boundedness of f by the boundedness of

$$\int_0^{2\pi} |f(re^{i\theta})|\, d\theta.$$

Solution: The proof is the same as for Liouville's theorem except that we estimate $|f(z) - f(0)|$ by

$$\frac{|z|}{2\pi(R - |z|)} \int_0^{2\pi} |f(Re^{i\theta})|\, d\theta.$$

15. Prove that if f is analytic in $|z| < 1$ and $|f(z)| \le 1 - |z|$, then $f(z) = 0$.

Solution: If $\rho > 1$, we have $|f(\rho e^{i\theta})| \le 1 - \rho$. For any given z with $|z| < \rho$, Cauchy's formula gives

$$|f(z)| \le \frac{1}{2\pi} \int_0^{2\pi} \frac{(1-\rho)\rho}{\rho - |z|}\, d\theta$$

Fixing z and letting $\rho \to 1$ we find that $|f(z)| = 0$; hence $f(z) = 0$.

16. Is it true that an analytic function f satisfies $|f(z)| \ge \dfrac{1}{1-|z|}$ for $|z| < 1$?

Solution: Suppose that there is such an f. If $|f(z)| > \dfrac{1}{1-|z|}$, then $f(z) \ne 0$. Thus $1/f(z)$ satisfies the hypothesis of Exercise and hence is identically zero. This is a contradiction.

17. Assume that $|f(z)| \le M$ on the circumference of a square whose side is L. Let z_0 be the centre of the square. If f is analytic in the square, prove that

$$|f'(z_0)| \le \frac{8M}{\pi L}.$$

Solution: There is no loss of generality if we take $z_0 = 0$. Then

$$f'(0) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{\zeta^2}$$

where C is the boundary of the square. The length of C is $4L$, and $|\zeta| \ge \dfrac{L}{2}$ on C, so

$$|f'(0)| \le \frac{1}{2\pi} 2LM \frac{4}{L^2} = \frac{8M}{\pi L}.$$

18. Expand in a Laurent series the function

$$f(z) = \frac{1}{z^2(z-1)} \text{ about } z = 0 \text{ and } z = 1.$$

Solution: $\dfrac{1}{z-1} = -\dfrac{1}{1-z} = -1 - z - z^2 - \ldots$

and $\dfrac{1}{z^2(z-1)} = -\dfrac{1}{z^2} - \dfrac{1}{z} - 1 - z \ldots$

$$\frac{1}{z^2} = \frac{1}{[1-(1-z)]^2}$$
$$= [1 + (1-z) + (1-z)^2 + \ldots]^2$$
$$= 1 + 2(1-z) + 3(1-z)^2 + \ldots$$
$$= 1 - 2(z-1) + 3(z-1)^2 - \ldots$$

and $$\frac{1}{z^2(z-1)} = \frac{1}{(z-1)} - 2 + 3(z-1) - 4(z-1)^2 + \ldots$$

19. Prove that the **Riemann zeta function** ζ defined by

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$$

converges for Re $z > 1$ and converges uniformly for Re $z \geq 1 + \varepsilon$ where $\varepsilon > 0$ is arbitrary small. Prove also that ζ is analytic for Re $z \geq 1 + \varepsilon$.

Solution:

$$\left|\frac{1}{n^z}\right| = \left|\frac{1}{e^{z \log n}}\right| = \left|\frac{1}{e^{x \log n}}\right| \left|\frac{1}{e^{iy \log n}}\right|$$
$$= \frac{1}{e^{x \log n}} = \frac{1}{n^x} \leq \frac{1}{n^{1+\varepsilon}}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}$ converges, we find by the Weierstrass *M* test that $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges uniformly for Re $z \geq 1 + \varepsilon$. We also see that each term of the series $\sum_{n=1}^{\infty} \frac{1}{n^z}$ is analytic function and since $\sum_{n=1}^{\infty} \frac{1}{n^z}$ is uniformly convergent for Re $z \geq 1 + \varepsilon$. Hence by Theorem 1 of Chapter 7,

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

is analytic for Re $z \geq 1 + \varepsilon$.

20. Expand the function $f(z) = \log(z+2)$ in a power series and determine its radius of convergence.

Solution: We have

$$f'(z) = \frac{1}{z+2} = \frac{1}{2}\frac{1}{1+\frac{z}{2}}$$
$$= \frac{1}{2}\left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \ldots\right)$$

Integrating it term by term we obtain

$$f(z) = C + \frac{1}{2}z - \frac{1}{4}\frac{z^2}{2} + \frac{1}{8}\frac{z^3}{3} - \ldots$$

Using $f(0) = \log 2$, we find $C = \log 2$.

The radius of convergence is given by

$$\frac{1}{R} = \lim \sqrt[n]{\frac{1}{2^n} \cdot \frac{1}{n}} = \frac{1}{2} \lim \sqrt[n]{\frac{1}{n}}$$

i.e. $R = 2 \lim \sqrt[n]{n} = 2.$

21. Prove that

(*i*) $\operatorname{Res}(\tan z, \pi/2) = -1$

(*ii*) $\operatorname{Res}\left(\frac{1-\cos z}{z^3}, 0\right) = \frac{1}{2}$

Solution:

(*i*) $$\tan z = \frac{\sin z}{\cos z} = \frac{\cos(\pi/2 - z)}{\sin(\pi/2 - z)}$$

At $z = \pi/2$, $\cos(\pi/2 - z) = 1$ and $\sin(\pi/2 - z)$ has a simple zero. Hence $z = \pi/2$ is a simple pole of $\tan z$. Thus

$$\operatorname{Res}(\tan z, \pi/2) = \lim_{z \to \pi/2} (z - \pi/2) \frac{\cos(\pi/2 - z)}{\sin(\pi/2 - z)} = -1$$

(*ii*) $$\frac{1-\cos z}{z^3} = \frac{z^2/2! - z^2/4! + \ldots}{z^3}$$

$$= \frac{1}{2!\,z} - \frac{z}{4!} + \ldots$$

Hence $$\operatorname{Res}\left(\frac{1-\cos z}{z^3}, 0\right) = \frac{1}{2}.$$

22. Prove that

$$\cot z = \frac{1}{z} + 2z \sum_{m=1}^{\infty} \frac{1}{z^2 - m^2\pi^2}$$

Solution:

Let $$f(z) = \cot z - \frac{1}{z} = \frac{z\cos z - \sin z}{z \sin z}$$

$f(z)$ has simple poles where

$$\sin z = 0,\ i.e.\ z = m\pi,\ m = \pm 1, \pm 2, \pm 3, \ldots$$

Also, $$\lim_{z \to 0} \frac{z\cos z - \sin z}{z \sin z} = 0$$

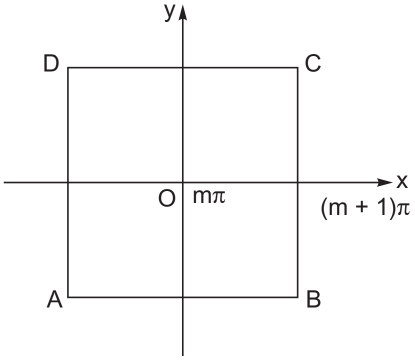
Thus we define $f(0) = 0$

$$\operatorname{Res}(f(z), m\pi) = \lim_{z \to m\pi} (z - m\pi) \frac{z\cos z - \sin z}{z \sin z} = 1$$

Using earlier results we obtain

$$\cot z = \frac{1}{z} + 2z \sum_{m=1}^{\infty} \frac{1}{z^2 - m^2\pi^2}$$

It remains to show that $f(z)$ is bounded on the contour Γ as shown in the figure.



ABCD is a square with center as origin and having sides of length $2m\,\pi + \pi$. The poles $\pm\,\pi, \pm\,2\pi, ..., \pm\,m\pi$ lie within the contour. Along BC $x = \left(m + \frac{1}{2}\right)\pi$. Now

$$|\cot z| = \left| i\frac{e^{2iz}+1}{e^{2iz}-1} \right| = \left| \frac{-e^{-2y}+1}{-e^{-2y}-1} \right| < 1.$$

Along AD, $x = -\left(m + \frac{1}{2}\right)\pi$, the same result holds.

Along CD, $$y = \left(m + \frac{1}{2}\right)\pi$$

and $$|\cot z| \le \left| \frac{1 + \exp[-(2m\pi + \pi)]}{1 - \exp[-(2m\pi + \pi)]} \right|$$

$$\to 1 \text{ as } m \to \infty$$

Along AB, similarly

$$|\cot z| \to 1 \text{ as } m \to \infty$$

Also $$\frac{1}{z} \to 0 \text{ on } \Gamma$$

Hence $$f(z) = \left(\cot z - \frac{1}{z}\right)$$

is bounded on Γ.

23. If β is not an integer, prove that

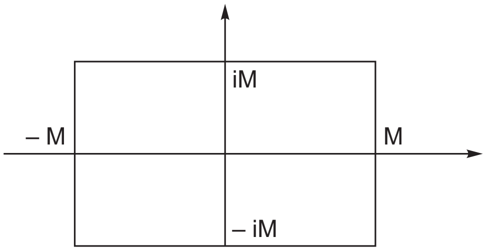
$$\sum_{n=-\infty}^{\infty} \frac{1}{\beta^2 - n^2} = \frac{\pi \cot \pi\beta}{\beta}$$

Solution: Consider

$$J = \frac{1}{2\pi i}\int_\gamma \frac{\pi \cot \pi z}{\beta^2 - z^2}\, dz$$

where γ is the boundary of the rectangular domain

$-\left(M + \frac{1}{2}\right) < x < M + \frac{1}{2}, -M < y < M$ and $M \ge 2$ is an integer (see the figure below).



The residue of $\dfrac{\pi \cot \pi z}{\beta^2 - z^2}$ at $z = n$, an integer is $\dfrac{1}{\beta^2 - n^2}$. If $|\beta| < M$, it follows from the residue theorem that

$$J = \sum_{n=-M}^{M} \frac{1}{\beta^2 - n^2} - \frac{\pi \cot \pi\beta}{\beta}$$

Note that the last term arises from the residue at – β and β. We have proved in the above exercise 22 that

$$|\cot \pi z| \le \cot h\, \pi,\ z \text{ on } \gamma$$

The length of γ is $8M + 2$, and since $|z| \ge M$ on γ, we find that

$$|J| \le \frac{8M+2}{2\pi} \cdot \frac{\pi \cot h\, \pi}{M^2 - |\beta|^2},\ M \ge 2,\ |\beta| < \text{M}.$$

This tends to 0 as $M \to \infty$ and hence the results follows.

24. Evaluate $\displaystyle\int_0^\infty \frac{x^2\, dx}{(x^2+1)^2}$.

Solution: We have

$$\int_0^\infty \frac{x^2}{(1+x^2)^2}\, dz = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2}\, dx$$

We will use the formula

$$\int_{-\infty}^{\infty} R(x)\, dx = 2\pi i \sum_{Im\, s_j > 0} \text{Res}\,(R, s_j)$$

where s_j are the poles of $R(z)$.

The function $R(z) = \dfrac{z^2}{(z^2+1)^2}$ has the only pole of order 2 at $z = i$ in the upper half plane. Hence $R(z)$ can be expressed in Laurent series about $z = i$, *i.e.*

$$R(z) = \frac{z^2}{(1+z^2)^2} = \frac{a_{-2}}{(z-i)^2} + \frac{a_{-1}}{(z-i)} + a_0 + a_1 (z-i) + a_2(z-i)^2 + \ldots$$

so that

$$R(z)\,(z-i)^2 = a_{-2} + a_{-1}\,(z-i) + a_0\,(z-i)^2 + a_1\,(z-i)^3 + \ldots$$

is analytic at $z = i$.

$$a_{-1} = \frac{d}{dz}[R(z)\,(z-i)^2]_{z=i}$$

$$= \frac{d}{dz}\left[\frac{z^2}{(z+1)^2}\right]_{z=i}$$

$$= \frac{1}{4i} = -\frac{i}{4}$$

Hence $$\int_{-\infty}^{\infty} R(x)\,dx = -2\pi i\left(\frac{i}{4}\right) = \pi/2$$

Thus $$\int_0^{\pi} \frac{x^2}{(1+x^2)^2}\,dx = \pi/4$$

25. Prove that

(*i*) $$\int_0^{2\pi} \frac{d\theta}{1+\sin^2\theta} = \pi\sqrt{2}$$

(*ii*) $$\int_0^{\pi} \frac{d\theta}{(1+1/2\cos\theta)^2} = 8\pi/3\sqrt{3}$$

Solution: (*i*) Let $z = e^{i\theta}$, $dz = iz\,d\theta$

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\int_0^{2\pi} \frac{d\theta}{1+\sin^2\theta} = \int_{|z|=1} \frac{dz}{iz\left[1 - \frac{1}{4}\left(z - \frac{1}{z}\right)^2\right]}$$

$$= -i\int_{|z|=1} \frac{dz}{z\left[1 - \frac{1}{4}\left(z^2 - 2 + \frac{1}{z^2}\right)\right]}$$

$$= i\int_{|z|=1} \frac{4z\,dz}{z^4 - 6z^2 + 1}$$

The simple poles of $\frac{4z}{z^4 - 6z^2 + 1}$ are

$$z_1 = \sqrt{2} - 1,\ z_2 = 1 - \sqrt{2},\ z_3 = \sqrt{2} + 1 \text{ and } z_4 = -1 - \sqrt{2},$$

But z_1 and z_2 are inside $|z| = 1$.

Hence $$\int_{|z|=1} \frac{4z}{z^4 - 6z^2 + 1}\,dz = 2\pi i \sum_{z_j \in |z|=1} \text{Res}\left(\frac{4z}{z^4 - 6z^2 + 1}, z_j\right)$$

Now $$\text{Res}\left(\frac{4z}{z^4 - 6z^2 + 1}, z_1\right) = \lim_{z \to z_1} (z - z_1)\frac{4z}{z^4 - 6z^2 + 1} = -\frac{1}{2\sqrt{2}}$$

Similarly $$\text{Res}\left(\frac{4z}{z^4 - 6z^2 + 1}, z_2\right) = -\frac{1}{2\sqrt{2}}$$

Hence $$i\int_{|z|=1} \frac{4z}{z^4 - 6z^2 + 1} = i2\pi i\left(-\frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right) = \pi\sqrt{2}$$

(*ii*) cos θ is an even function. Hence

$$I = \int_0^{\pi} \frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2} = \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2}$$

Setting $z = e^{i\theta}$, we find

$$\frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2} = \frac{1}{2}\int_{|z|=1} \frac{dz}{iz\left[1+\frac{1}{4}\left(z+\frac{1}{z}\right)\right]^2}$$

$$= -8i\int_{|z|=1} \frac{z}{z^4+8z^3+18z^2+8z+1}\,dz$$

$$= -8i\int_{|z|=1} \frac{z}{(z^2+4z+1)^2}\,dz$$

The poles of $\frac{z}{(z^2+4z+1)^2}$ are $z_1 = \sqrt{3}-2$ and $z_2 = -\sqrt{3}-2$, both of order 2. z_1 is inside $|z| = 1$ but z_2 is outside $|z| = 1$.

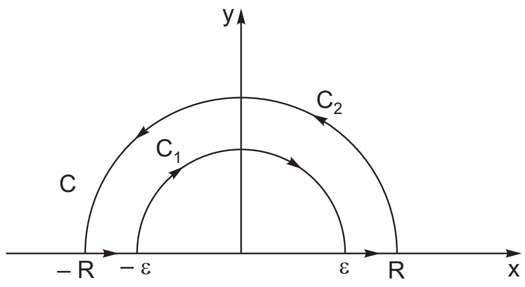
$$\text{Res}\left(\frac{z}{(z^2+4z+1)}, z_1\right) = \sqrt{3}/18$$

Hence
$$I = -8i\,2\pi i \sum_{z_j\in|z|=1} \text{Res}\left(\frac{z}{(z^2+4z+1)^2}, z_j\right)$$

$$= 16\pi\sqrt{3}/18 = 8\pi/3\sqrt{3}$$

26. Prove that $\int_0^{\infty} \frac{(\log x)^2}{1+x^2}\,dx = \pi^3/8$

Solution: Let C be the curve as shown in the figure here and let C_1 and C_2 be the semi-circles of radii ε and R respectively with centre at the origin.



Consider

$$\int_C \frac{(\log z)^2}{z^2+1}\,dz$$

$\frac{(\log z)^2}{z^2+1}$ has a simple pole $z = i$ which lies inside C

$$\operatorname{Res}\left(\frac{(\log z)^2}{z^2+1}, i\right) = \lim_{z\to i}(z-i)\frac{(\log z)^2}{(z+i)(z-i)} = -\pi^2/8i$$

Hence $$\int_C \frac{(\log z)^2}{z^2+1}dz = 2\pi i\left(-\frac{\pi^2}{8i}\right) = -\frac{\pi^3}{4}$$

Now $$\int_C \frac{(\log z)^2}{z^2+1}dz = \int_{-R}^{-\epsilon}\frac{(\log z)^2}{z^2+1}dz + \int_{C_1}\frac{(\log z)^2}{z^2+1}dz$$
$$+\int_{\varepsilon}^{R}\frac{(\log z)^2}{z^2+1}dz + \int_{C_2}\frac{(\log z)^2}{z^2+1}dz$$

Setting $z = -x$ in the first integral and $z = x$ in the third integral on the right, we find

$$\int_{\varepsilon}^{R}\frac{(\log x+\pi i)^2}{x^2+1}dx + \int_{C_1}\frac{(\log z)^2}{z^2+1}dz + \int_{\varepsilon}^{R}\frac{(\log x)^2}{x^2+1}dx$$
$$+\int_{C_2}\frac{(\log z)^2}{z^2+1}dz = -\pi^3/4$$

Let $\varepsilon\to 0$ and $R\to\infty$. It can be easily checked that the integrals around C_1 and C_2 approach zero. Hence

$$\int_0^{\infty}\frac{(\log x+\pi i)^2}{x^2+1}dx + \int_0^{\infty}\frac{(\log x)^2}{x^2+1}dx = -\frac{\pi^3}{4}$$

i.e. $$2\int_0^{\infty}\frac{(\log x)^2}{x^2+1}dx + 2\pi i\int_0^{\infty}\frac{\log x}{x^2+1}dx - \pi^2\int_0^{\infty}\frac{dx}{x^2+1} = -\frac{\pi^3}{4}$$

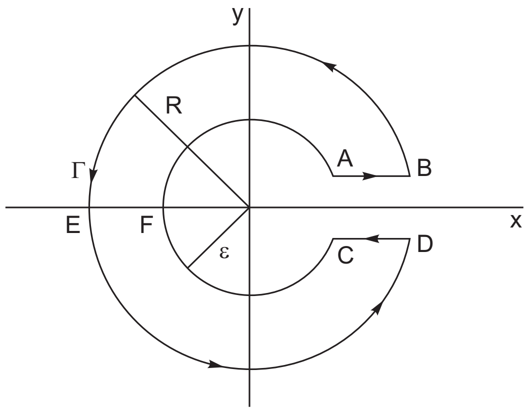
i.e. $$2\int_0^{\infty}\frac{(\log x)^2}{x^2+1}dx + 2\pi i\int_0^{\infty}\frac{\log x}{x^2+1}dx - \frac{\pi^3}{2} = -\frac{\pi^3}{4}$$

i.e. $$2\int_0^{\infty}\frac{(\log x)^2}{x^2+1}dx + 2\pi i\int_0^{\infty}\frac{\log x}{x^2+1}dx = \frac{\pi^3}{4}$$

Hence $$2\int_0^{\infty}\frac{(\log x)^2}{x^2+1}dx = \frac{\pi^3}{8}$$

27. Prove that

$$\int_0^{\infty}\frac{x^a}{x(1+x)}dx = \frac{\pi}{\sin \pi a}, \ (0 < a < 1)$$



Solution: Consider

$\int_\Gamma \frac{z^{a-1}}{1+z}\,dz$ where Γ is the contour indicated in the figure shown here. Note that $z = 0$ is a branch point and hence the contour Γ is chosen in this fashion. $\frac{z^{a-1}}{1+z}$ has the simple pole $z = -1$ which is inside Γ.

$$\text{Res}\left(\frac{z^{a-1}}{1+z}, e^{\pi i}\right) = \lim_{z\to -1} (z+1)\frac{z^{a-1}}{1+z} = e^{(a-1)\pi i}$$

Hence $$\int_\Gamma \frac{z^{a-1}}{1+z}\,dz = 2\pi i\, e^{(a-1)\pi i}$$

i.e. $$\int_{AB} + \int_{BED} + \int_{DC} + \int_{CFA} = 2\pi i\, e^{(a-1)\pi i}$$

i.e. $$\int_0^R \frac{x^{a-1}}{1+x}dx + \int_0^{2\pi} \frac{(Re^{i\theta})^{a-1} iRe^{i\theta} d\theta}{1+Re^{i\theta}}$$

$$+ \int_R^{\varepsilon} \frac{(xe^{2\pi i})^{a-1} dx}{1+xe^{2\pi i}} + \int_{2\pi}^{0} \frac{(\varepsilon e^{i\theta})^{a-1} i\varepsilon e^{i\theta} d\theta}{1+\varepsilon e^{i\theta}} = 2\pi i\, e^{(a-1)\pi i}$$

Let $\varepsilon\to 0$ and $R\to\infty$. It can be easily checked that the second and fourth integrals tend to zero. Thus

$$\int_0^\infty \frac{x^{a-1}}{1+x}dx + \int_\infty^0 \frac{e^{2\pi i(a-1)} . x^{a-1}}{1+x}dx = 2\pi i\, e^{(a-1)\pi i}$$

i.e. $$\{1 - e^{2\pi i(a-1)}\}\int_0^\infty \frac{x^{a-1}}{1+x}dx = 2\pi i\, e^{(a-1)\pi i}$$

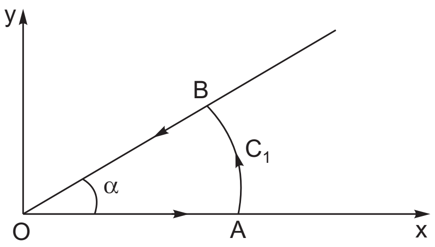
Hence $$\int_0^\infty \frac{x^{a-1}}{1+x}dx = \frac{2\pi i}{e^{a\pi i} - e^{-a\pi i}} = \frac{\pi}{\sin a\pi}$$

28. Prove that

(*i*) $$\int_0^\infty e^{-x^2 \cos 2\alpha} \cos(x^2 \sin 2\alpha)\,dx = \frac{\sqrt{\pi}}{2}\cos\alpha$$

(*ii*) $$\int_0^\infty e^{-x^2 \cos 2\alpha} \sin(x^2 \sin 2\alpha)\,dx = \frac{\sqrt{\pi}}{2}\sin\alpha$$

Solution: Consider the contour Γ as shown in the figure here.



The contour Γ consists of the line OA, the arc C_1 of the circle $|z| = r$ from A to B, $(0 \le \alpha \le \pi/4)$ and the line BO.

$$(1)\quad \int_{\Gamma} e^{-z^2}\,dz = \int_0^r e^{-x^2}\,dx + \int_{C_1} e^{-z^2}\,dz + \int_r^0 \exp\{-(\rho e^{i\alpha})^2\} e^{i\alpha}\,d\rho$$

(on BO, $z = \rho e^{i\alpha}$, $0 \le \rho \le r$)

Since e^{-z^2} is analytic within and on contour Γ, we have

$$\int_{\Gamma} e^{-z^2}\,dz = 0$$

It can be checked that

$$\int_{C_1} e^{-z^2}\,dz \to 0 \text{ as } r \to \infty$$

Hence, letting $r \to \infty$, we obtain from (1)

$$\int_0^{\infty} \exp(-\rho^2 e^{2i\alpha})\, e^{i\alpha}\,d\rho = \int_0^{\infty} e^{-x^2}\,dx = \sqrt{\pi}/2$$

i.e.

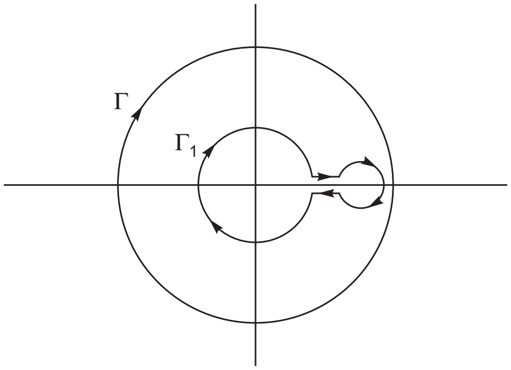
$$(2)\quad \int_0^{\infty} e^{-\rho^2 \cos 2\alpha}\,[\cos(\rho^2 \sin 2\alpha) - \sin(\rho^2 \sin 2\alpha)]\,2\rho = \sqrt{\pi}/2e^{-i\alpha}$$

Equating real and imaginary parts on the two sides of (2) and replacing the variable of integration ρ by x, we get (*i*) and (*ii*).

Note that, putting $\alpha = \pi/4$ in (*i*) and (*ii*) we obtain

$$\int_0^{\infty} \cos(x^2)\,dx = \int_0^{\infty} \sin(x^2)\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

29. Prove that $\displaystyle\int_0^1 \frac{dx}{(x^2 - x^3)^{1/3}} = \frac{2\pi}{\sqrt{3}}$



Solution: Let Γ and Γ_1 be the contours as shown in the figure here.

$z = 0$ and $z = 1$ are the branch points of $\dfrac{1}{(z^2 - z^3)^{1/3}}$.

We have

$$\int_{|z|=r} f(z)\,dz = \int_{2\pi}^{0} \frac{re^{i\theta} i\,d\theta}{(r^2 e^{2i\theta} - r^3 e^{3i\theta})^{1/3}} \to 0 \text{ as } r \to 0$$

and $$\int_{|z-1|=\rho} f(z)\,dz = \int_{\pi}^{-\pi} \frac{\rho e^{i\theta} i\, d\theta}{(1+\rho e^{i\theta})^{2/3}(-\rho e^{i\theta})^{1/3}} \to 0 \text{ as } \rho \to 0$$

Here $f(z)$ is analytic for every finite z except on the cut from 0 to 1. Hence

$$\int_{\Gamma} f(z)\,dz - \int_{\Gamma_1} f(z)\,dz = 0$$

The Residue* of $f(z)$ at $z = \infty$ is

$$\lim_{z\to\infty} \{-z\,f(z)\} = \lim_{z\to\infty} \frac{-z}{z\left(\frac{1}{z}-1\right)^{1/3}} = -(-1)^{-1/3}$$

Hence $$\int_{\Gamma} f(z)\,dz = -2\pi i\,(-1)^{-1/3}$$

But $$\int_{\Gamma} f(z)\,dz = \int_{\Gamma} f(z)\,dz = (1-e^{2\pi i/3})\int_0^1 \frac{dx}{(x^2-x^3)^{1/3}}$$

$$= -2\pi i\,(-1)^{-1/3}$$

Therefore

$$\int_0^1 \frac{dx}{(x^2-x^3)^{1/3}} = \frac{2\pi i}{e^{\pi i/3}-e^{-\pi i/3}} = \frac{2\pi}{\sqrt{3}}$$

30. Find the number of zeros of the function

$F(z) = \sin z + 2iz^2$ in the rectangle

$|x| \le \pi/2,\ |y| \le 1.$

Solution: Apply Rouche's Theorem with $2iz^2$ as f and $\sin z$ as g. For $z = x \pm i$,

$$|\sin z| = (\sin^2 x \cosh^2 1 + \cos^2 x \sinh^2 1)^{1/2}$$
$$< \cosh 1 < 1.5431$$

For $z = \pi/2 + iy$, $|y| \le 1$, $|\sin z| = \cosh y \le \cosh 1$. But $2|z^2| \ge 2$. Hence $|\sin z| < |2iz^2|$ on the boundary of the rectangle, and there are two zeros inside. Note that one zero is at 0.

31. A Möbius transformation that transforms the real axis to the real axis can be represented with real coefficients a, b, c, d.

We assume that none of the coefficients is zero. Let $w = \dfrac{az+b}{cz+d}$. Note that the inverse image of 0 is real, so b/a is real; the image of ∞ is real, so d/c is real. Now we represent the transformation as

$$w = \frac{(a/b)z+1}{(c/d)z+1} = \frac{lz+1}{nz+1}$$

where l and n are real.

---

*Let $w = f(z)dz$ where f is a rational function. Write $t = 1/z$. The residue of w at ∞ is defined to be the residue of $-t^2 f(1/t)$ at $t = 0$.

32. Find the Möbius transformation which maps the unit disc of the z-plane into the unit disc of the w-plane.

Let $$w = \frac{az+b}{cz+d}.$$

Observe that $w = 0$ and $w = \infty$ correspond to symmetric points α and $1/\overline{\alpha}$ where $|\alpha| < 1$. Then $b = -a\alpha$, $c = -\overline{\alpha}d.$ It follows that

$$w = \left(-\frac{a}{d}\right)\frac{(z-\alpha)}{(\overline{\alpha}z-1)}.$$

But when $|z| = 1$, $\left|\frac{(z-\alpha)}{(\overline{\alpha}z-1)}\right| = 1$ and therefore

$$-\frac{a}{d} = e^{i\theta} \quad (\theta \text{ real})$$

Hence $$w = e^{i\theta}\frac{z-\alpha}{\overline{\alpha}z-1}, |\alpha| < 1, \theta \text{ real}.$$

33. Find the Möbius transformation which maps the upper half of the z-plane into the upper half of the w-plane.

We have seen above that Möbius transformation that maps the real axis to the real axis can be represented with real coefficients. Conversely, a transformation with real coefficients maps the real axis to the real axis. Hence, the Möbius transformation in this case must be

$$w = \frac{az+b}{cz+d} \text{ where } a, b, c, d \text{ are real.}$$

In order to make the upper half-plane correspond to the upper half-plane, we have to make $z = i$ correspond to a point w with $\operatorname{Im} w > 0$. Then

$$\frac{ai+b}{ci+d} = \frac{ac+bd-bci+adi}{c^2+d^2}$$

must have positive imaginary part, so $ad - bc > 0$.

34. Find the Möbius transformation which maps the right-hand half of the z-plane into the right-hand half of the w-plane.

In order to transform the right-hand half plane to the right-hand half plane, we first transform the right-hand half plane to the upper-half ζ-plane by

$$\zeta = iz$$

Then $$w = \frac{a\zeta+b}{c\zeta+d}$$

transforms the upper half plane to the upper half plane, and multiplication by $-i$ carries this back to the right-hand half plane. Thus,

$$w = -i\,\frac{aiz+b}{ciz+d} = \frac{az-bi}{ciz+d},\ ad-bc > 0.$$

Hence, the required Mobius transformation is

$$w = i\,\frac{az-bi}{cz-di}.$$

35. Prove that if f is analytic and Re $f < 0$ in the disk $|z| \le 1$, then

$$|f'(0)| \le -2 \operatorname{Re} f(0).$$

Solution: Let $f(0) = A$. Then $w = f(z)$ maps the z disk $|z| < 1$ into the left-hand w-plane, and 0 to A. Now we map the half plane to the unit disk in a t-plane by

$$t = g(w) = \frac{A - w}{\overline{A} + w}.$$

Then $g\,|f(z)|$ maps the unit disk into the unit disk and 0 to 0. By Schwarz's lemma (see Appendix I),

$$\left|\frac{d}{dz} g\, f(z)\right| \le 1$$

i.e. $\quad |\{g'[f(0)]f'(0)\}_{z=0}| \le 1$

But $$g'(w) = \frac{-(\overline{A} + w) - (A - w)}{(\overline{A} + w)^2} = \frac{-(\overline{A} + A)}{(\overline{A} + w)^2},$$

$$g'(A) = -\frac{1}{\overline{A} + A},\ A = f(0).$$

It follows that

$$\left|\frac{f'(0)}{\overline{A} + A}\right| \le 1$$

$$|f'(0)| \le |\overline{A} + A| = |2\, Re f(0)| = -2\, Re f(0).$$

36. Let f be analytic in $|z| \le 1$. Suppose that $0 < |f(z)| < 1$ in $|z| \le 1$. Then prove that

$$|f'(0)| \le 2/e$$

Solution: Since $f(z) \ne 0$ in $|z| \le 1$, there is an analytic branch of $\psi(z) = \log f(z)$ in $|z| \le 1$. Then Re $\psi(z) < 0$ because $|f(z)| < 1$. It follows from Exercise that

$$|\psi'(0)| \le -2 \log |f(0)|,$$

$$\frac{|f'(0)|}{|f(0)|} \le -2 \log |f(0)|,$$

$$|f'(0)| \le -2\,|f(0)| \log |f(0)|.$$

The maximum of $-t \log t$ for $0 < t < 1$ is $1/e$ at $t = 1/e$. Hence

$$|f'(0)| \le \frac{2}{e}$$

37. Find the image of $-\pi/2 + k\pi \le \operatorname{Re} z \le \pi/2 + k\pi$ (k = Constant) under the mapping

$$\omega = \sin z$$

Solution: Write the mapping $\omega = \sin z$ into three mappings:

$$\omega = \sin z = \frac{e^{iz} - e^{-iz}}{2} f \circ g \circ h$$

where $h = iz$, $g = e^h$ and $f = \dfrac{1}{2i}\left(g - \dfrac{1}{8}\right)$.

Since $h = iz$ is a rotation by $\pi/2$, the image of the strip

$$ST_1 = \{z : -\pi/2 + k\pi \le \operatorname{Re} z \le \pi/2 + k\pi\}$$

under the map $h = iz$ is again a strip

$$ST_2 = \{z : -\pi/2 + k\pi \le \text{Im}\, z \le \pi/2 + k\pi\}.$$

The image of the strip ST_2 under the map $g = e^h$ is the half plane $H \sim \{0\}$, where

$$H = \{z : -\pi/2 + k\pi \le arg\, z \le \pi/2 + k\pi\}$$

Now it remains to find the image of $H \sim \{0\}$ by f.

Note that if k is an even integer,

$H = \{z : -\pi/2 \le arg\, z \le \pi/2\}$ and if k is an odd integer,

$H = \{z : -\pi/2 \le arg\, z \le 3\pi/2\}$

Setting $f = u + iv$ and $z = re^{i\theta}$, we have

$$f(z) = \frac{1}{2i}\left(z - \frac{1}{z}\right) = -\frac{i}{2}\left(re^{i\theta} - \frac{1}{r}r^{-i\theta}\right)$$

Hence $$u = \left(r + \frac{1}{r}\right)\frac{\sin\theta}{2}, \quad v = \left(r - \frac{1}{r}\right)\frac{\cos\theta}{2}$$

It follows that

$$\frac{u^2}{\sin^2\theta} - \frac{v^2}{\cos^2\theta} = 1.$$

i.e. the image of the ray $\{re^{i\theta}$; θ is constant$\}$ is the hyperbola whose axes are $\sin\theta$ and $\cos\theta$.

Also, $$\left(\frac{2u}{r + \frac{1}{r}}\right)^2 + \left(\frac{2v}{\frac{1}{r} - r}\right)^2 = 1,$$

i.e. the image of a circle $|z| = r$ is the ellipse with axes

$$\frac{1}{2}\left(r + \frac{1}{r}\right), \frac{1}{2}\left(\frac{1}{r} - r\right).$$

When k is an even integer, v is positive for $0 < r < 1$ but v is negative for $r > 1$; u is positive for $0 \le \theta \le \pi/2$ and negative for $-\pi/2 \le \theta < 0$.

Thus, when k is an even integer, the image of $H \cap D'(0, 1)$ under the map f is the upper half plane $\text{Im}\, z > 0$ and the image of $H - D'(0, 1)$ is the lower half plane $\text{Im}\, z \le 0$.

Hence, the image of the strip ST_1 with k even under the map $w = \sin z$ is the entire complex plane C.

38. Prove that $\log \dfrac{1+z}{1-z}$ is an isomorphism of disc $D(0, 1)$ with the strip $-\pi/2 < \text{Im}\, z < \pi/2$.

Solution: $z_1 = 1 - z$ is an isomorphism

$$D(0, 1) \to D(1, 1);$$

$$z_2 = \frac{1}{z_1} \text{ is an isomorphism}$$

$$D(1, 1) \to \text{Re}\,(z_2) > \frac{1}{2};$$

$$z_3 = -1 + 2z_2 \text{ is an isomorphism}$$

$$\text{Re}\,(z_2) > \frac{1}{2} \to \text{Re}\,(z_3) > 0;$$

Write $z_4 = \log z_3 = \log |z_3| + i$

arg z_3 is an isomorphism

$$\text{Re}\,(z_3) > 0 \to -\pi/2 < \text{Im}\,(z_4) < \pi/2$$

Hence

$$z_4 = \log z_3 = \log\,(1 + 2z_2) = \log\left(1 + \frac{2}{z_1}\right)$$

$$= \log\left(1 + \frac{2}{1-z}\right) = \log\left(\frac{1+z}{1-z}\right)$$

is an isomorphism

$$D(0, 1) \to \{z : -\pi/2 < \text{Im}\, z < \pi/2\}$$

39. Find the image of the given domain D under the following conformal mappings

(*i*) $w = z + \dfrac{1}{z}$; $D : |z| < 1$, Im $z > 0$

(*ii*) $w = \dfrac{1 - \cos z}{1 + \cos z}$; $D : 0 < \text{Re}\, z < \pi/2$

Solution:

(*i*) Setting $w = u + iv$ and $z = re^{i\theta}$, we find

$$u = \left(r + \frac{1}{r}\right)\cos\theta \text{ and } v = \left(r - \frac{1}{r}\right)\sin\theta$$

Now Im $z > 0$ iff $0 < \theta < \pi$ and $|z| < 1$ iff $r < 1$. Thus $\sin\theta > 0$ and $r - \dfrac{1}{r} < 0$ implies $v < 0$ with v taking all negative values. $u \geq 0$ for $0 < \theta \leq \pi/2$ and $u \leq 0$ for $\pi/2 \leq \theta < \pi$, u taking all real values.

Hence, $w = z + \dfrac{1}{z}$ maps the region $(|z| < 1) \cap (\text{Im}\, z > 0)$ onto the lower half-plane Im $z < 0$.

(*ii*) Write the mapping $w = \dfrac{1 - \cos z}{1 + \cos z} = -\left(\dfrac{e^{iz} - 1}{e^{iz} + 1}\right)^2$ as the composition of mappings :

$$z_1 = e^{iz},\ z_2 = \frac{z_1 - 1}{z_1 + 1} \text{ and } z_3 = -z_2^{\,2} = w$$

arg z_1 = arg e^{iz} = arg $e^{ix-y} = x = \text{Re}\, z$.

Thus $0 < \text{Re}\, z < \pi/2$ implies $0 < \arg z_1 < \pi/2$.

It can be easily checked that Re $z > 0$ is mapped onto $|z_2| < 1$ by

$$z_2 = \frac{z_1 - 1}{z_1 + 1}$$

and it is an isomorphism of $0 < \arg z_1 < \pi/2$ onto the region

$$(| z_2 | < 1) \cap (\operatorname{Im} z_2 > 0)$$

or $$(| z_2 |) < 1) \cap (0 < \arg z_2 < \pi)$$

Since $| z_3 | = | z_2 |^2$ and $\arg z_3 = \pi + 2 \arg z_2$ it follows that

$$0 < \arg z_2 < \pi \text{ iff } \pi < \arg z_3 < 3\pi \text{ and } | z_2 | < 1 \text{ iff } | z_3 | < 1$$

Finally, we find that the mapping

$$w = \frac{1 - \cos z}{1 + \cos z} \text{ maps } 0 < \operatorname{Re} z < \pi/2 \text{ onto}$$

$$(| w | < 1) \cap (-\pi < \arg w < \pi)$$

40. Let $z_1, z_2, z_1 \neq z_2$, be the two fixed points of a Möbius transformation T. Show that T may be written as:

$$\frac{w - z_1}{w - z_2} = k\frac{z - z_1}{z - z_2}$$

where k is a complex constant.

Show that the cross ratio of z_1, z_2, z, w is constant.

Solution: Let z_1 and z_2 be the fixed points. Then $z_1 = T(z_1)$ and $z_2 = T(z_2)$.

Let $T_1 = \frac{z - z_1}{z - z_2}$. T_1 maps $z_1 \to 0$ and $z_2 \to \infty$ and

$$T_1 \circ T \circ T_1^{-1} \text{ maps } 0 \to 0 \text{ and } \infty \to \infty.$$

Hence $T_1 \circ T \circ T_1^{-1}$ is given by

$$T_1 \circ T \circ T_1^{-1}(u) = ku$$

It follows that $T = T_1^{-1} \circ ku \circ T_1$

i.e. $$\frac{w - z_1}{w - z_2} = k\frac{z - z_1}{z - z_2} \text{ defines } T.$$

We will consider the particular case of the second part. We will determine z_1, z_2 and k for the transformation

$$w = \frac{5z - 2}{z + 3}$$

Its fixed points are given by

$$z = \frac{5z - 2}{z + 3}.$$

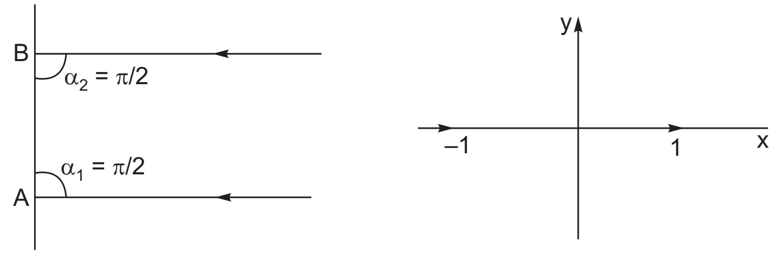
i.e. $$z = 1 \pm i$$

Hence $$\frac{w - (1 + i)}{w - (1 - i)} = \frac{\frac{5z-2}{z+3} - (1 + i)}{\frac{5z-2}{z+3} - (1 - i)}$$

$$= \frac{4 - i}{4 + i} \cdot \frac{z - \frac{5 + 3i}{4 - i}}{z - \frac{5 - 3i}{4 + i}}$$

Thus $$k = \frac{15 - 8i}{17}$$

41. Find a transformation which maps the semi-infinite strip onto the upper half plane.



Solution:

Here $\alpha_1 = \pi/2 = \alpha_2$ as shown in the figure here. We choose $x_1 = 1$ and $x_2 = -1$ and apply the Schwarz-Christoffel transformation. Thus we have

$$\frac{dw}{dz} = K\,(z+1)^{(\pi/2)/\pi-1}\,(z-1)^{(\pi/2)/\pi-1}$$
$$= K\,(z+1)^{-1/2}\,(z-1)^{-1/2}$$

Hence $$w = K\int \frac{dz}{\sqrt{(z^2-1)}} = K\cosh^{-1} z + D$$

where D is a constant.

To determine K and D, we have $z = 1$ for $w = 0$, this gives $D = 0$ and $z = -1$ for $w = \pi i$, this gives $K = 1$. Thus

$$w = \cosh^{-1} z.$$

42. Let f be analytic in $|z| \le 1$. Suppose that $|f(z)|$ is maximized for $|z| \le 1$ at z_0 with $|z_0| = 1$. Prove that $f'(z_0) \ne 0$ unless f is constant.

Solution: Suppose that $f'(z_0) = 0$. Then in a neighbourhood $U : |z - z_0| < \varepsilon$ of z_0, (ε is small) f is analytic in U. Now f can be represented by Taylor series with centre at z_0. We have, for $f \ne 0$,

$$f(z) - f(z_0) = (z - z_0)^m \left[a_m + \sum_{n=m+1}^{\infty} a_n (z - z_0)^{n-m}\right] \ (a_m \ne 0,\ m \ge 2).$$

When ε is small, the part of U inside $|z| = 1$ is nearly half a disk, and arg z varies by nearly π on the part E of ∂U which is inside $|z| = 1$. It follows that arg $[f(z) - f(z_0)]$ varies by nearly $k\pi$ on $E (m \ge 2)$, and this makes $w = f(z)$ go outside the disk $|w| \le f(z_0)|$. Hence, $|f|$ takes values greater than $|f(z_0)|$ at some points z with $|z| < 1$. This contradicts the hypothesis that $|f|$ has its maximum at z_0 for $|z| \le 1$.

43. Suppose that U and V are conjugate harmonic functions and let

$$\phi(x) = \frac{V(x, y)}{U(x, y)}$$

Prove that

$$\phi(x) = \tan(\alpha x + \beta)$$

where α and β are real numbers. Also, determine $U(x, y)$ and $V(x, y)$.

Solution:

Since U and V are conjugate harmonic functions, hence

$$f(z) = U(x, y) + iV(x, y) \text{ is analytic and}$$

$$\frac{V(x, y)}{U(x, y)} = \tan(\arg f(z))$$

i.e. $$\arg f(z) = \text{arc} \tan\left(\frac{V}{U}\right)$$

$$= \text{arc} \tan \phi(x)$$

Now $$\arg f(z) = \text{Im} \log f(z)$$

Hence $$\frac{\partial}{\partial y}[\text{Im} \log f(z)] = 0$$

Since $\log f(z)$ is differentiable when $f(z) \neq 0$, it follows by $C - R$ equations that

$$\frac{\partial}{\partial x} \text{Re} \log f(z) = \frac{\partial}{\partial y} \text{Im} \log f(z) = 0$$

Therefore, $$\text{Re} \log f(z) = g(y)$$

and $$g'(y) = \frac{\partial}{\partial y} g(y) = -\frac{\partial}{\partial x} \text{Im} \log f(z)$$

$$= -\frac{\partial}{\partial x} \text{arc} \tan \phi(x)$$

Since $g'(y)$ depends only on y, and $-\frac{\partial}{\partial x}$ arc tan $\phi(x)$ only on x, we find that

$$g'(y) = -\frac{\partial}{\partial x} \text{arc} \tan \phi(x) = -\alpha \text{ (a real constant).}$$

Thus $$\text{arc} \tan \phi(x) = \alpha x + \beta$$

i.e. $$\tan(\alpha x + \beta) = \phi(x)$$

It remains to determine U and V.

Since $$g(y) = -\alpha y + \delta$$

and $$g(y) = \text{Re} \log f(z) = \log |f(z)|$$

we have $$|f(z)| = e^{g(y)} = e^{-\alpha y + \delta}$$

We know that $$f(z) = |f(z)| \text{ cis} \arg f(z)$$

Hence

$$f(z) = e^{-\alpha y + \delta} \text{ cis } (\alpha x + \beta) = e^{\delta} \text{ cis } \beta e^{-\alpha y} \text{ cis } \alpha x$$

Thus $$U(x, y) = Ce^{-\alpha y} \cos \alpha x$$

and $$V(x, y) = Ce^{-\alpha y} \sin \alpha x$$

where $$C = e^{\delta} \text{ cis } \beta$$

44. Find the conjugate function of

$$u = x^2 - y^2$$

Solution:

The function $u = x^2 - y^2$ is harmonic. Also, $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = -2y$. Hence a conjugate of u will satisfy

(3) $$\frac{\partial v}{\partial y} = 2x, \frac{\partial v}{\partial x} = 2y$$

Integrating the first equation of (3) with respect to y we have

(4) $v = 2xy + k(x)$ where $k(x)$ depends only on x.

Substituting (4) in the last equation of (3) we obtain $k'(x) = 0$ and so $k(x) =$ constant $= C$.

Hence, the conjugate function of $x^2 - y^2$ is $2xy + C$ and the analytic function $f(z)$ is given by

$$f(z) = (x^2 - y^2) + i(2xy + C) = z^2 + iC.$$

45. Prove that the harmonic conjugate of $\frac{x}{x^2+y^2}$ is $\frac{-y}{x^2+y^2}$.

Solution: $$\frac{x}{x^2+y^2} = \text{Re}\left(\frac{1}{z}\right)$$

We have $$f(z) = \frac{1}{z}$$

is analytic in C except at the origin *i.e.* $z = 0$.

Threfore the real part of $\frac{1}{z}$ is harmonic in C except at $z = 0$. But the harmonic conjugate of Re $\left(\frac{1}{z}\right)$ is Im $\left(\frac{1}{z}\right)$. That is, the harmonic conjugate of $\frac{x}{x^2+y^2}$ is $\frac{-y}{x^2+y^2}$.

46. Prove that under the transformation $w = f(z)$ is harmonic function $\phi(x, y)$ remains harmonic where $f(z)$ is analytic and $f'(z) \neq 0$.

Solution: Let $\phi(x, y)$ be transformed into a function $\phi[x(u, v), y(u, v)]$ by the transformation. We will first prove that

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = |f'(z)|^2\left(\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}\right)$$

We have $$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial\phi}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial u}\right) + \frac{\partial\phi}{\partial v}\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial v}\right)$$

$$= \frac{\partial\phi}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\left[\frac{\partial^2\phi}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2\phi}{\partial v\,\partial u}\frac{\partial v}{\partial x}\right]$$

$$+ \frac{\partial\phi}{\partial v}\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\left[\frac{\partial^2\phi}{\partial u\,\partial v}\frac{\partial u}{\partial x} + \frac{\partial^2\phi}{\partial v^2}\frac{\partial v}{\partial x}\right]$$

Similarly
$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial \phi}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\left[\frac{\partial^2 \phi}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 \phi}{\partial v \partial u}\frac{\partial v}{\partial y}\right] + \frac{\partial \phi}{\partial v}\frac{\partial^2 v}{\partial y^2} + \frac{\partial v}{\partial y}\left[\frac{\partial^2 \phi}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 \phi}{\partial v^2}\frac{\partial v}{\partial y}\right]$$

Adding,

$$(5) \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial \phi}{\partial u}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\partial^2 \phi}{\partial u^2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + 2\frac{\partial^2 \phi}{\partial u \partial v}\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right] + \frac{\partial^2 \phi}{\partial v^2}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

Now, by $C - R$ equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. Also, since u and v are harmonic,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Then
$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$
$$= \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = |f'(z)|^2$$

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0.$$

Hence (5) reduces to

$$(6) \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = |f'(z)|^2\left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right)$$

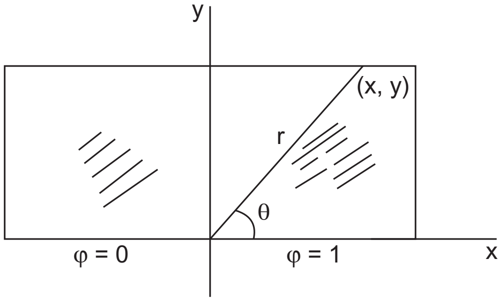
It follows from (6) that if $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial v^2} = 0$ and $f'(z) \neq 0$ then

$$\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} = 0$$

47. Determine a function which is harmonic in the upper half plane, Im $(z) > 0$ and which takes the prescribed values

$$F(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

Solution: This is a Dirichlet problem for the upper half plane as shown in the figure



In other words, we have to solve for $\phi\,(x, y)$ the boundary-value problem

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0,\ y > 0\ ;$$

$$\lim_{y\to 0+} \phi(x, y) = F(x) = \begin{cases} 1 & x > 0 \\ 0 & x > 0 \end{cases}$$

Since $A\theta + B$ is the imaginary part of $A \log z + B$ where A and B are real constants, hence $A\theta + B$ is harmonic.

We have to determine A and B under the conditions

$$\varphi = 1 \text{ for } x > 0$$

$$\varphi = 0 \text{ for } x < 0$$

Thus

(*i*) $1 = A(0) + B$, (*ii*) $0 = A(\pi) + B$

Hence $\quad A = -\,1/\pi,\ B = 1$ and $\phi = 1 - \dfrac{1}{\pi}\tan^{-1}\left(\dfrac{y}{x}\right)$

One can also solve this type of problem by using Poisson's formula for the half plane.

48. Let $\psi(\theta) = 0$ for $0 < \theta < \pi$ and 1 for $\pi < \theta < 2\pi$. Find a series for the function $u(r, \theta)$ which is harmonic in the unit disk with these boundary values. Using the series compute $u\,(1/2, \pi/2)$ numerically.

Solution:

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{0} e^{-in\theta}\,d\theta,\ a_0 = \frac{1}{2}$$

For $n \neq 0$,

$$a_n = -\frac{1}{2\pi\,ni}[1 - (-1)^n]$$

$$\psi(\theta) = \frac{1}{2} - \frac{1}{\pi i}\sum_{\text{odd } n=1}\frac{1}{n}e^{in\theta}$$

$$= \frac{1}{2} - \frac{1}{\pi i}\sum_{\text{odd } n=1}\frac{1}{n}(e^{in\theta} - e^{-in\theta})$$

$$= \frac{1}{2} - \frac{2}{\pi}\sum_{\text{odd } n}\frac{\sin n\,\theta}{n}$$

Hence, $$u(r, \theta) = \frac{1}{2} - \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{\sin(2m+1)\theta}{(2m+1)} r^{2m+1}$$

$$u\left(\frac{1}{2}, \frac{\pi}{2}\right) = \frac{1}{2} - \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m+1}(2m+1)}$$

$$= \frac{1}{2} - \frac{2}{\pi} \tan^{-1}(1/2)$$

$$= 0.2048\ldots .$$

49. Prove that if $|z| < \frac{1}{2}$, then

(i) $|\log(1+z) - z| \leq |z|^2$,

(ii) $|\log(1+z) - z + \frac{1}{2}z^2| \leq |z|^3$,

(iii) $|\log(1+z) - z + \frac{1}{2}z^2 - \frac{1}{3}z^3| \leq |z|^4$,

(iv) $|\log(1+z) - z + \frac{1}{2}z^2 - \frac{1}{3}z^3 + \frac{1}{4}z^4 || \leq || z|^5$,

and so on.

Solution: If $|z| < \frac{1}{2}$,

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots$$

Hence $$|\log(1+z) - z| \leq |z|^2 \left| \frac{1}{2} - \frac{z}{3} + \frac{z^2}{4} - \ldots \right|$$

$$\leq |z|^2 \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right)$$

$$= |z|^2$$

This proves (i). We can prove similarly (ii), (iii), (iv). We verify (ii) below

$$\left|\log(1+z) - z + \frac{z^2}{2}\right| \leq |z|^3 \left| \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \right|$$

$$= |z|^3$$

50. Prove that π $(1 + a_n)$ converges if both $\Sigma\, a_n$ and $\Sigma\, |a_n|^2$ converge.

Solution: It follows from the previous exercise that

$$|\log(1 + a_n) - a_n| \leq |a_n|^2.$$

Hence, if $\Sigma\, |a_n|^2$ converges, so does $\Sigma\, [\log(1 + a_n) - a_n]$; and then if $\Sigma\, a_n$ converges, so does $\Sigma \log(1 + a_n)$.

51. Investigate the convergence of π $(1 + a_n)$ in the following cases:

(i) $a_n = (-1)^n n^{-1/3}$

(ii) $a_{2n+1} = (2n-1)^{-1/2}$, $a_{n+2} = -(2n)^{-1/2}$, $n = 1, 2, 3, \ldots$

(*iii*) $a_n = 2/n$, $n = 1, 2, 3, \ldots$

(*iv*) $a_n = -\dfrac{z^2}{n^2}$

Solution: (*i*) We have

$$|\log(1 + a_n) - a_n + \frac{a_n^2}{2} - \frac{a_n^3}{3}| < |a_n|^4 = \frac{1}{n^{4/3}},$$

so
$$\Sigma\left[\log(1 + a_n) - a_n + \frac{a_n^2}{2} - \frac{a_n^3}{3}\right]$$

converges absolutely. Now $\Sigma\, a_n$ and $\Sigma\, a_n^3$ converge (conditionally), but Σa_n^2 diverges and therefore $\Sigma \log(1 + a_n)$ diverges.

(*ii*) Since $\Sigma\, |a_n^3|$ converges, it follows that

$$\Sigma\left[\log(1 + a_n) - a_n + \frac{a_n^2}{2}\right] \text{ converges}$$

According as n is odd or even,

$$a_n + \frac{a_n^2}{2} = \frac{1}{n^{1/2}} - \frac{1}{2n} \text{ or } -\frac{1}{(n-2)^{1/2}} - \frac{1}{2(n-2)},$$

so
$$\Sigma\left(a_n + \frac{1}{2}a_n^2\right) \text{ diverges, hence } \Sigma \log(1 + a_n) \text{ diverges.}$$

(*iii*) Since $|\log(1 + a_n) - a_n| \le \dfrac{|z|^2}{n^2}$, it follows that $\Sigma\,[\log(1 + a_n) - a_n]$ converges. But since $\Sigma\, a_n$ diverges, the product diverges.

(*iv*) Since $\Sigma\, |a_n|$ converges and $\Sigma\,[\log(1 + a_n) - a_n]$ converges, it follows that the product converges.

52. Prove that $\prod\limits_{n=1}^{\infty}(1 - z^2/n^2)$ converges uniformly on each compact set that excludes all the points ± n.

Solution: Let z belong to a compact set E that contains no point $\pm n$. Since E is bounded, it follows that for sufficiently large n, $|z^2/n^2| < \dfrac{1}{2}$ and

$$|\log\left(1 - \frac{z^2}{n^2}\right) + \frac{z^2}{n^2}| \le \frac{1}{2}|\frac{z^2}{n^2}|^2 \le \frac{k}{n^4}$$

for some number k. Since $\Sigma\, z^2/n^2$ converges uniformly on E, it follows that $\Sigma \log(1 - z^2/n^2)$ converges uniformly on E.

53. Prove **Stirling's formula**

$$\lim_{n \to \infty}(n!\, e^n n^{-n-1/2}) = \sqrt{2}\pi$$

Solution: Let $a_n = \log(n!\, e^n n^{-n-1/2})$

Then

$$a_n = (n + 1/2)\ \{\log (n + 1/2) - \log n\} - \frac{1}{2} + \sum_{m=1} \log m - \int_0^{n+1/2} \log x dx$$

Setting $f(x) = \log x$ in the following identity

$$\int_0^{n+1/2} f(x)\, dx = \int_0^{1/2} \left[f(x) + \sum_{m=1}^{n} \{f(m + x) + f(m - x)\} \right] dx$$

we obtain

$$(7) \qquad \int_0^{n+1/2} \log x\, dx - \sum_{m=1}^{n} \log m = \int_0^{1/2} \left\{ \log x + \sum_{m=1}^{n} \log\left(1 - \frac{x^2}{m^2}\right) \right\} dx$$

But

$$\sum_{m=1}^{\infty} \log\left(1 - \frac{x^2}{m^2}\right)$$

converges uniformly for $0 \le x \le \frac{1}{2}$.

Taking the limit in both sides of (7), we have

$$(8) \qquad \lim_{n\to\infty} \left(\int_0^{n+1/2} \log x\, dx - \sum_{m=1}^{n} \log m \right)$$

$$= \int_0^{1/2} \left\{ \log x + \sum_{m=1}^{\infty} \log\left(1 - \frac{x^2}{m^2}\right) \right\} dx$$

Using the result

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right),$$

the right side of (8) becomes

$$\int_0^{1/2} \log \sin \pi\, x\, dx - \frac{1}{2} \log \pi$$

It can be easily checked that

$$\int_0^{1/2} \log \sin \pi x\, dx > \frac{1}{2} \log 2$$

It now follows that

$$\lim_{n\to\infty} a_n = \frac{1}{2} \log (2\pi)$$

i.e.

$$\lim_{n\to\infty} (n!\, e^n\, n^{-n-1/2}) = \sqrt{2}\pi.$$

Define the **Gamma function** by the formula

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^2}{z(z+1)(z+2)\ldots(z+n)}$$

From the definition it follows that $\Gamma(z)$ does not exist if z is zero or a negative integer.

54. Prove that $\Gamma(z)$ exists for all other values of z other than zero and the negative integers.

Solution: Let Ω_0 be the set of all (finite) points other than zero and the negative integers. Let

$$(9) \qquad \Gamma_n(z) = \frac{n!\,n^z}{z(z+1)\ldots(z+n)}$$

$$(10)\ g_n(z) = \log(n!) + z\log n - \sum_{m=0}^{n} \log(z+m)$$

where $z \in \Omega_0$ and n is a natural number.

Then

$$(11) \qquad \Gamma_n(z) = \exp g_n(z)$$

and

$$(12) \qquad \Gamma(z) = \lim_{n\to\infty} \Gamma_n(z)$$

Let $z \in \Omega_0$. Then, for any natural number n,

$$g_n(z) = g_1(z) + \sum_{m=2}^{n} \{g_m(z) - g_{m-1}(z)\}$$

Now, if m is an integer, and $m > \max(1, |z|)$, then

$$g_m(z) - g_{m-1}(z) = \log m + z\{\log m - \log(m-1)\} - \log(z+m)$$

$$= \int_0^1 \frac{z(z+1)t}{(m-t)(m+tz)}\,dt$$

Hence, if $m > \max(1, |z|)$

$$|g_m(z) - g_{m-1}(z)| \le |z(z+1)|(m-1)^{-1}(m - |z|^{-1})$$

It now follows that

$$\sum_{m=2}^{\infty} \{g_m(z) - g_{m-1}(z)\}$$

converges and hence $\lim_{n\to\infty} g_n(z)$ exists. Thus $\Gamma(z)$ exists for all $z \in \Omega_0$.

Define the **beta** function by

$$(13) \qquad B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1}\,dt$$

55. Prove that

$$B(z, w+1) = B(z, w)\cdot w/(z+w)$$

for Re $z > 0$ and Re $w > 0$.

Solution: $(z+w)B(z, w+1) - wB(z, w)$

$$= (z+w)\int_0^1 t^{z-1}(1-t)^w\,dt - w\int_0^1 t^{z-t}(1-t)^{w-1}dt$$

$$= \int_0^1 \{zt^{z-1}(1-t)^w - t^z w(1-t)^{w-1}\}\,dt$$

$$= [t^z(1-t)^w]_0^1 = 0$$

56. Prove that

$$\Gamma_n (z) = n^z B (z, n + 1) \text{ for } \operatorname{Re} z > 0 \text{ and } n \text{ is a natural number.}$$

Solution: $B (z, n + 1) = \dfrac{n}{z + n} B (z, n)$

$$= \frac{n}{z+n} \frac{n-1}{z+n-1} \cdots \frac{1}{z+1} B (z, 1)$$

But $\qquad B (z, 1) = \displaystyle\int_0^1 t^{z-1}\, dt = \frac{1}{z}$

Hence, the result follows from (10).

Observe that

(14) $\qquad \displaystyle\lim_{n \to \infty} \{n^z \; B (z, w + n + 1) = \Gamma (z).$

57. Prove that

$$\lim_{n \to \infty} \{n^z B (z, w \; + n + 1)\} = \Gamma (z) \text{ for } \operatorname{Re} z > 0$$

and n is a natural number.

Solution: For Re $w \geq 1$,

$$| n^2 B (z, w + n + 1) - n^z B (z, n + 1) |$$

$$= n^{\operatorname{Re} z} \left| \int_0^1 t^{z-1} \{(1-t)^w - 1\} (1-t)^n \; dt \right|$$

$$\leq n^{\operatorname{Re} z} \int_0^1 t^{\operatorname{Re} z - 1} |(1-t)^w - 1| (1-t)^n \; dt,$$

and $\qquad | (1 - t)^w - 1 | = \left| \displaystyle\int_0^t w (1 - x)^{w-1} dx \right|$

$$\leq t \, | w | \; (0 \leq t \leq 1)$$

Hence, by (13) we have

(15) $\quad | n^z B (z, w + n + 1) - n^z B (z, n + 1) | \leq | w | \, n^{\operatorname{Re} z} B (\operatorname{Re} z + 1, n + 1)$

It now follows from (14) that

$$\lim_{n \to \infty} \{n^{\operatorname{Re} z + 1} B (\operatorname{Re} z + 1, n + 1)\} = \Gamma (\operatorname{Re} z + 1)$$

Hence, the right side of (15) tends to zero as $n \to \infty$.

This proves the result under the restriction Re $w \geq 1$. But this restriction is not necessary as can be seen from the following argument.

Consider z fixed for a moment

Then $\displaystyle\lim_{n \to \infty} \{n^n B (z, w + n + 1)\} = g (w)$ (say)

Now whenever $g (w)$ exists, $g (w) = g (w - k)$ for any positive integer k. This shows that Re $w \geq 1$ is not necessary.

58. Prove that

$$B (z, w) = \Gamma (z) \, \Gamma (w) / \Gamma (z + w)$$

for $\qquad$ Re $z > 0$ and Re $w > 0$

Solution: Let $F(w) = B(z, w)\,\Gamma(z + w)/\Gamma(w)$

Considering z fixed, we have $F(w) = F(w + 1)$ and $F(w) = F(w + n + 1)$. It can be checked from the previous results that

$$\begin{aligned} F(w) &= \lim_{n \to \infty} F(w + n + 1) \\ &= \lim_{n \to \infty} \left\{ n^z\, B(z, w + n + 1) \frac{\Gamma(z + w + n + 1)}{n!\, n^{z+w}} \frac{n!\, n^w}{\Gamma(w + n + 1)} \right\} \\ &= \Gamma(z) \end{aligned}$$

59. Prove that

$$\Gamma(z) = \int_0^\infty u^{z-1} e^{-u} du \text{ for } \operatorname{Re} z > 0$$

Solution: Since $e^x \geq 1 + x$ and $e^{-x} \geq 1 - x$ for every real number x, it follows that

(16) $1 \geq e^x(1 - x) \geq (1 + x)(1 - x) = 1 - x^2$ $(x < 1)$.

Now let n be a natural number. Then, by (16) if $u \leq n$, then

$$1 \geq e^{u/n}\left(1 - \frac{u}{n}\right) \geq \left(1 - \frac{u^2}{n^2}\right)$$

and hence

$$(17) \qquad 1 \geq e^u\left(1 - \frac{u}{n}\right)^n \geq \left(1 - \frac{u^2}{n^2}\right)^n \qquad (0 \leq u \leq n)$$

Now $(1 - x)^n \geq 1 - nx$ $(x \leq 1)$. Hence,

$$1 \geq e^u\left(1 - \frac{u}{n}\right)^n \geq 1 - \frac{u^2}{n} \qquad (0 \leq u \leq n),$$

so that

$$(18) \qquad 0 \leq e^{-u} - \left(1 - \frac{u}{n}\right)^n \leq e^{-u}\frac{u^2}{n} \qquad (0 \leq u \leq n)$$

Using (13), we have

$$\begin{aligned} n^z B(z, n + 1) &= \int_0^1 (nt)^{z-1}(1 - t)^n\, n dt \\ &= \int_0^n u^{z-1}\left(1 - \frac{u}{n}\right)^n du \end{aligned}$$

It can be checked easily that

$$\int_0^\infty u^{z-1} e^{-u}\, du \text{ exists if and only if } \operatorname{Re} z > 0$$

Thus it follows from (18) that

$$\begin{aligned} &\left| \int_0^\infty u^{z-1} e^{-u}\, du - n^z B(z, n + 1) \right| \\ &\leq \int_0^n u^{\operatorname{Re} z - 1}\left\{ e^{-u} - \left(1 - \frac{u}{n}\right)^n \right\} du + \int_n^\infty u^{\operatorname{Re} z - 1} e^{-u} du \end{aligned}$$

$$\leq n^{-1}\int_0^{\infty} u^{\operatorname{Re} z+1}\, e^{-u} du$$

Hence, the result follows.

60. Let $G = \{z : \operatorname{Re} z \geq a\}$ where $a > 1$. Prove that if $\varepsilon > 0$ then there is a number δ, $0 < \delta < 1$, such that for all z in G.

$$(19) \qquad \left| \int_{\alpha}^{\beta} (e^t - 1)^{-1}\, t^{z-1}\, dt \right| < \varepsilon$$

whenever $\delta > \beta > \alpha$.

Solution: Since $e^t - 1 \geq t$ for all $t \geq 0$ then for $0 < t \leq 1$ and z in G

$$|\, (e^t - 1)^{-1}\, t^{z-1} \,| \leq t^{a-2}$$

Since $a > 1$ the integral $\int_0^t t^{a-2}\, dt$ is finite. Hence δ can be found to satisfy (19).

61. Let $F = \{z : \operatorname{Re} z \leq A\}$ where $-\infty < A < \infty$. Prove that if $\in\, > 0$ then there is a number $k > 1$ such that for all z in F

$$\left| \int_{\alpha}^{\beta} (e^t - 1)^{-1}\, t^{z-1} dt \right| < \varepsilon$$

whenever $\beta > \alpha > k$.

Solution: If $t \geq 1$ and z is any point in F then there is a constant C such that

$$|\, (e^t - 1)^{-1}\, t^{z-1} \,| \leq (e^t - 1)^{-1}\, t^{A-1} \leq Ce^{1/2t}\, (e^t - 1)^{-1}$$

Since $e^{1/2t}\, (e^t - 1)^{-1}$ is integrable on $[1, \infty]$ the required number k can be found.

Proceeding similarly we can obtain the following results.

62. Let $H = \{z : a \leq \operatorname{Re} z \leq A\}$ where $1 < a < A < \infty$ then the integral

$$\int_0^{\infty} (e^t - 1)^{-1}\, t^{z-1} dt$$

converges uniformly on H.

63. If $E = \{z : \operatorname{Re} z \leq A\}$ where $-\infty < A < \infty$ then the integral

$$\int_1^{\infty} (e^t - 1)^{-1}\, t^{z-1}\, dt$$

converges uniformly on E.

64. Prove that

$$\zeta(z)\, \Gamma(z) = \int_0^{\infty} (e^t - 1)^{-1}\, t^{z-1}\, dt \text{ for } \operatorname{Re} z > 1.$$

Solution: It follows from the above results that this integral is an analytic function in the region $\{z : \operatorname{Re} z > 1\}$. Thus it is sufficient to prove that $\zeta(z)\, \Gamma(z)$ equals this integral for $z = x > 1$.

It follows from exercises 60 and 61 above that there are numbers α and β, $0 < \alpha < \beta < \infty$, such that

$$\int_0^{\alpha} (e^t - 1)^{-1}\, t^{x-1}\, dt < \varepsilon/4,$$

$$\int_{\beta}^{\infty} (e^t - 1)^{-1}\, t^{x-1}\, dt < \varepsilon/4$$

Since $$\sum_{k=1}^{n} e^{-kt} \le \sum_{k=1}^{\infty} e^{-kt} = (e^t - 1)^{-1}$$

for all $n \ge 1$,

$$\sum_{n=1}^{\infty} \int_0^{\alpha} e^{-nt}\, t^{x-1}\, dt < \varepsilon/4,$$

$$\sum_{n=1}^{\infty} \int_{\beta}^{\infty} e^{-nt}\, t^{x-1}\, dt < \varepsilon/4$$

Using the relation $$\zeta(z)\, \Gamma(z) = \sum_{n=1}^{\infty} n^{-z}\, \Gamma(z)$$

$$= \sum_{n=1}^{\infty} \int_0^{\infty} e^{-nt}\, t^{z-1}\, dt$$

we have $$\left| \zeta(x)\, \Gamma(x) - \int_0^{\infty} (e^t - 1)^{-1}\, t^{x-1}\, dt \right|$$

$$\le \varepsilon + \left| \sum_{n=1}^{\infty} \int_{\alpha}^{\beta} e^{-nt}\, t^{x-1}\, dt - \int_{\alpha}^{\beta} (e^t - 1)^{-1}\, t^{x-1}\, dt \right|$$

But $\Sigma\, e^{-nt}$ converges to $(e^t - 1)^{-1}$ uniformly on $[\alpha, \beta]$, so the result is proved.

By using the above result we would like to extend the domain of definition of ζ to $\{z : \operatorname{Re} z > -1\}$.

Let us consider the Laurent expansion of the function

$$\text{(20)} \qquad \frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} a_n z^n$$

where $a_1, a_2, \ldots$ are some constants. The function $\left[\dfrac{1}{e^t - 1} - \dfrac{1}{t}\right]$ is bounded in a neighbourhood of $t = 0$. This implies that

$$\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) t^{z-1}\, dt$$

converges uniformly on compact subsets of $\{z : \operatorname{Re} z > 0\}$ and hence represents an analytic function on the right half plane. Thus

$$\text{(21)} \qquad \zeta(z)\, \Gamma(z) = \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) t^{z-1}\, dt + \frac{1}{z-1} + \int_1^{\infty} \frac{t^{z-1}}{e^t - 1}\, dt.$$

By using exercises 62 and 63 we see that each of these summands except $\dfrac{1}{z-1}$, is analytic in $\{z : \operatorname{Re} z > 0\}$. Thus we may define $\zeta(z)$ for $\operatorname{Re} z > 0$ in the following manner.

$$\zeta(z) = \frac{1}{\Gamma(z)} \left[\int_0^t \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) t^{z-t}\, dt + \frac{1}{z-1} + \int_1^{\infty} \frac{t^{z-1}}{e^t - 1}\, dt \right]$$

Hence ζ is meromorphic in the right half plane with a simple pole at $z = 1$ whose residue is 1. Let $0 < \operatorname{Re} z < 1$, then

$$\frac{1}{z-1} = -\int_1^\infty t^{z-2}\,dt$$

Inserting this in (21) we obtain

$$\text{(22)} \qquad \zeta(z)\,\Gamma(z) = \int_0^\infty \left(\frac{1}{e^{t-1}} - \frac{1}{t}\right) t^{z-1}\,dt,\ 0 < \operatorname{Re} z < 1.$$

In fact, our aim is to establish **Riemann's Functional Equation.**

65. $\zeta(z) = 2\,(2\pi)^{z-1}\,\Gamma(1-z)\,\zeta(1-z)\sin\left(\frac{1}{2}\pi z\right)$ for $-1 < \operatorname{Re} z < 0$.

Considering again the Laurent expansion of $\frac{1}{e^z - 1}$ we find that

$$\left[\frac{1}{(e^t-1)} - \frac{1}{t} + \frac{1}{2}\right] \le Ct$$

for some constant C and all t in $[0, 1]$. Thus

$$\int_0^1 \left(\frac{1}{e^{t-1}} - \frac{1}{t} + \frac{1}{2}\right) t^{z-1} dt$$

is uniformly convergent on compact subsets of $\{z : \operatorname{Re} z > -1\}$. Also, since

$$\lim_{t\to\infty} t\left(\frac{1}{e^t-1} - \frac{1}{t}\right) = 1$$

there is a constant C_1 such that

$$\left(\frac{1}{e^t-1} - \frac{1}{t}\right) \le \frac{C_1}{t},\ t \ge 1.$$

This implies that

$$\int_1^\infty \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1} dt$$

converges uniformly on compact subsets of $\{z : \operatorname{Re} z < 1\}$.

Using these results we find that

$$\text{(23)} \qquad \zeta(z)\,\Gamma(z) = \int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right) t^{z-1} dt - \frac{1}{2z} + \int_1^\infty \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1} dt \text{ for } 0 < \operatorname{Re} z < 1.$$

Since both integrals on the right side of (23) converge in the strip $-1 < \operatorname{Re} z < 1$, we can define $\zeta(z)$ in $\{z : -1 < \operatorname{Re} z < 1\}$.

The question naturally arises: What happens at $z = 0$? Since $1/2\,z$ appears on the right side of (23), is it true that ζ have a pole at $z = 0$? It can be easily checked that this is not true. Dividing both sides of (23) by $\Gamma(z)$ we will get one term

$$\frac{1}{2z\,\Gamma(z)} = \frac{1}{2\,\Gamma(z+1)}$$

which is analytic at $z = 0$.

Thus if ζ is so defined in the strip $\{z : -1 < \operatorname{Re} z < 1\}$ it is analytic there. Combining this with (21) we find that $\zeta(z)$ is defined for $\operatorname{Re} z > -1$ with a simple pole at $z = 1$. Now if $-1 < \operatorname{Re} z < 0$ then

$$-\frac{1}{z} = \int_1^\infty t^{z-1}\,dt$$

Inserting this in (23) we obtain

$$\zeta(z)\,\Gamma(z) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) t^{z-1}\,dt,\ -1 < \operatorname{Re} z < 0 \tag{24}$$

We have $\displaystyle \frac{1}{e^t - 1} + \frac{1}{2} = \frac{1}{2}\left(\frac{e^t + 1}{e^t - 1}\right) = \frac{i}{2}\cot\left(\frac{1}{2}it\right)$

$$\cot\left(\frac{1}{2}it\right) = \frac{2}{it} - 4\,it \sum_{n=1}^{\infty} \frac{1}{t^2 + 4n^2\pi^2} \quad \text{for } t \neq 0.$$

Thus $\displaystyle \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)\frac{1}{t} = 2\sum_{n=1}^{\infty} \frac{1}{t^2 + 4n^2\pi^2}$

Inserting this in (24) we get

$$\begin{aligned} \zeta(z)\,\Gamma(z) &= 2\int_0^\infty \left(\sum_{n=1}^{\infty} \frac{1}{t^2 + 4n^2\pi^2}\right) t^z\,dt \\ &= 2\sum_{n=1}^{\infty} \int_0^\infty \frac{t^z}{t^2 + 4n^2\pi^2}\,dt \\ &= 2\sum_{n=1}^{\infty} (2\pi n)^{z-1} \int_0^\infty \frac{t^z}{t^2 + 1}\,dt, \\ &= 2\,(2\pi)^{z-1}\,\zeta(1 - z) \int_0^\infty \frac{t^z}{t^2 + 1}\,dt, \end{aligned} \tag{25}$$

valid for $-1 < \operatorname{Re} z < 0$.

Substituting $s = t^2$ for the real x with $-1 < x < 0$ we have

$$\begin{aligned} \int_0^\infty \frac{t^z}{t^2 + 1}\,dt &= \frac{1}{2}\int_0^\infty \frac{s^{1/2(x-1)}}{s + 1}\,ds \\ &= \frac{1}{2}\pi \operatorname{cosec}\left[\frac{1}{2}\pi(1 - x)\right] \\ &= \frac{1}{2}\pi \sec\left(\frac{1}{2}\pi x\right) \end{aligned} \tag{26}$$

But

$$\begin{aligned} \frac{1}{\Gamma(x)} &= \frac{\Gamma(1 - x)}{\pi}\sin \pi x \\ &= \frac{\Gamma(1 - x)}{\pi}\, 2\sin\left(\frac{1}{2}\pi x\right)\cos\left(\frac{1}{2}\pi x\right) \end{aligned} \tag{27}$$

It follows from (24), (26) and (27) that

$$\zeta(z) = 2(2\pi)^{z-1}\,\Gamma(1-z)\,\zeta(1-z)\sin\left(\frac{1}{2}\pi z\right) \tag{28}$$

valid for $-1 < \operatorname{Re} z < 0$.

In fact, we established **Riemann's functional equation** for real x with $-1 < x < 0$. But since both sides of (28) are analytic on the strip $-1 < \operatorname{Re} z < 0$, (28) follows. Following similarly we can show that (28) holds for $-1 < \operatorname{Re} z < 1$.

66. If $\operatorname{Re} z > 1$ then

$$\zeta(z) = \prod_{n=1}^{\infty}\left(\frac{1}{1-p_n^{-z}}\right)$$

where $\{p_n\}$ is the sequence of **prime numbers.**

Solution: Using the geometric series we find that

$$\frac{1}{1-p_n^{-z}} = \sum_{m=0}^{\infty} p_n^{-mz} \tag{29}$$

for all $n \geq 1$. Taking the product of the terms $\left(\frac{1}{1-p_k^{-z}}\right)$ for $1 \leq k \leq n$ where $n \geq 1$ and using (29) we obtain

$$\prod_{k=1}^{n}\left(\frac{1}{1-p_n^{-z}}\right) = \sum_{j=1}^{\infty} n_j^{-z} \tag{30}$$

Note that the integers $n_1, n_2, n_3, \ldots$ are all the integers which can be factored as a product of powers of the prime numbers $p_1, p_2, \ldots, p_n$ alone. Allowing $n \to \infty$ we get the required result.

Weierstrass Elliptic Function $\mathcal{P}(z)$

We recall the following definitions. An analytic function f is called periodic if there exists a (complex) constant $w \neq 0$ such that

$$f(z+w) = f(z)$$

for all $z \in C$. The constant w is called a period of the function. If there does not exist an integer n such that w/n is also a period, then we call w a fundamental period. If $\operatorname{Im}(\overline{w}_1\, w_2) = 0$, where $w_1 \neq w_2$, then f is called simply periodic. Note that w_1 and w_2 are distinct periods. If $\operatorname{Im}(\overline{w}_1\, w_2) \neq 0$, where $w_1 \neq w_2$, then f is called doubly periodic.

The doubly periodic analytic functions are called elliptic functions.

Observe that if h is any analytic function for which the series

$$\sum_{-\infty}^{\infty} h(z - nw) = f(z)$$

converges uniformly, then f will be periodic with period w. If w is a sum of any integer multiples of w_1 and w_2 with $\operatorname{Im}(\overline{w}_1\, w_2) \neq 0$, then

$$\sum_{w} h(z-w) = p(z)$$

Note that the summation over all possible w yields $p(z)$, a doubly periodic function

Note also that

$$h\,(z) + \sum_{w \neq 0} [h\,(z - w) - h\,(w)] = p\,(z)$$

is doubly periodic.

Setting $h\,(z) = 1/z^2$ in the above expression we obtain the Weierstrass elliptic function

$$\frac{1}{z^2} + \sum_{w \neq 0} \left\{ \frac{1}{(z-w)^2} - \frac{1}{w^2} \right\} = \wp(z).$$

67. Show that $\wp(z)$ satisfies the differential equations

(31) $$6\,\wp^2 - \wp'' = 30\,k_4$$

and

(32) $$4\,\wp'' - \wp'^2 - 60\,k_4 = 140\,k_6$$

where $\wp'$ and $\wp''$ are the first and second derivatives of $\wp$ and

(33) $$k_4 = \sum_{w \neq 0} \frac{1}{w^4},\; k_6 = \sum_{w \neq 0} \frac{1}{w^6}$$

Solution: By definition

$$\wp(z) = \frac{1}{z^2} + \sum_{w \neq 0} \left\{ \frac{1}{(z-w)^2} - \frac{1}{w^2} \right\}$$

$$-\,\wp'\,(z) = 2\sum_{w} \frac{1}{(z-w)^3}$$

$$= 2\left\{ \frac{1}{z^3} + \sum_{w \neq 0} \frac{1}{(z-w)^3} \right\}$$

and

(34) $$\wp'\,(z) = 6\sum_{w} \frac{1}{(z-w)^4}$$

$$= 6\left\{ \frac{1}{z^4} + \sum_{w \neq 0} \frac{1}{(z-w)^4} \right\}$$

where the summation $\sum\limits_{w}$ includes the origin ($w = 0$).

Note that $\wp$ has a double pole at the origin and $\wp''$ has a fourth order pole at the origin. Thus the fourth-order pole of $\wp^2$ at the origin can be eliminated by subtracting $\wp''/6$ from $\wp^2$. We find

(35) $$\wp^2 = \frac{1}{z^4} + \frac{2}{z^2}\Sigma_2 + \Sigma_2^2$$

and for all small z

(36) $$\frac{1}{(z-w)^2} = \frac{1}{w^2}\left\{ 1 + 2\left(\frac{z}{w}\right) + 3\left(\frac{z}{w}\right)^2 \right\}$$

and $$\Sigma_2 = \sum_{w \neq 0} \left[\frac{1}{(z-w)^2} - \frac{1}{w^2} \right]$$

$$= 2z \sum_{w \neq 0} \frac{1}{w^2} + 3z^2 \sum_{w \neq 0} \frac{1}{w^4}$$

Since $\sum_{w \neq 0} \frac{1}{w^m} = 0$ for m = odd integer

(37) $\lim_{z \to 0} \Sigma_2 = 0$

(38) $\lim_{z \to 0} \frac{1}{z^2} \Sigma_2 = 3k_4$

$$\Sigma_2 = 3k_4 z^2$$

Hence, it follows from (34) and (35) that

(39) $6\,\wp^2 - \wp'' = \frac{12}{z^2} \Sigma_2 + 6\,\Sigma_2^2 - 6\,\Sigma_4$

where

(40) $\Sigma_4 = \sum_{w \neq 0} \frac{1}{(z-w)^4}$

Since $6\,\wp^2 - \wp''$ is an elliptic function it is in fact a constant if we could show the existence of

$$\lim_{z \to 0} [6\,\wp^2 - \wp''].$$

It follows from (37), (38) and (40) that the right side of (39) becomes

$$6\,\wp_0^2 - p''_0 = 36\,k_4 - 6\,k_4 = 30\,k_4$$

as $z \to 0$. Also since $z = 0$ is the only singularity of $6\,\wp^2 - \wp''$ we find that

$$6\,\wp^2 - \wp'' = 36\,k_4 \text{ for all } z$$

In order to prove (32) we need to multiply (31) by $2\wp'$ and after integrating we get

(41) $4\wp^3 - \wp'^2 - 60\,k_4\,\wp =$ constant

This constant can be determined by taking the limit of the left side of (41) as $z \to 0$

Writing

$$-\wp'(z) = 2\left\{\frac{1}{z^3} + \sum_{w \neq 0} \frac{1}{(z-w)^3}\right\} = 2\left\{\frac{1}{z^3} + \Sigma_3\right\}$$

we have

(42) $\wp'^2 = 4\left\{\frac{1}{z^6} + \frac{2}{z^3}\Sigma_3 + \Sigma_3^2\right\}$

Since $\wp(z) = \frac{1}{z^2} + \Sigma_2$, we have

(43) $4\,\wp^3 = 4\left\{\frac{1}{z^6} + \frac{3}{z^4}\Sigma_2 + \frac{3}{z^2}\Sigma_2^2 + \Sigma_2^3\right\}$

Thus (41) becomes

$$\frac{12}{z^4}\Sigma_2 - \frac{8}{z^3}\Sigma_3 - \frac{60\,k_4}{z^2} + \frac{12}{z^2}\Sigma_2^2 + 4\,\Sigma_2^3 - 4\,\Sigma_3^2 - 60\,k_4\,\Sigma_2$$

The last four terms of the above expression approach zero as $z \to 0$. The first two terms in the above expression can be evaluated as follows:

For small z

$$\frac{1}{(z-w)^2} \cong \frac{1}{w^2}\left\{1 + 2\left(\frac{z}{w}\right) + 3\left(\frac{z}{w}\right)^2 + 4\left(\frac{z}{w}\right)^3 + 5\left(\frac{z}{w}\right)^4\right\}$$

Hence

(44) $$\Sigma_2 = \sum_{w \neq 0}\left[\frac{1}{(z-w)^2} - \frac{1}{w^2}\right] \cong 3\,k_4\,z^2 + 5\,k_6\,z^4$$

Also, since $\displaystyle\sum_{w\neq 0}\frac{1}{w^3} = \sum_{w\neq 0}\frac{1}{w^5} = 0$

and $$\frac{1}{(z-w^3)} \cong -\frac{1}{w^3}\left\{1 + 3\left(\frac{z}{w}\right) + 6\left(\frac{z}{w}\right)^2 + 10\left(\frac{z}{w}\right)^3\right\}$$

Hence

(45) $$-\Sigma_3 = -\sum_{w\neq 0}\frac{1}{(z-w)^3} \cong 3k_4\,z + 10k_6\,z^3$$

Thus

$$\lim_{z\to 0}\left\{\frac{36k_4}{z^2} + 60k_6 + \frac{24k_4}{z^2} + 80k_6 - \frac{60k_4}{z^2}\right\} = 140k_6$$

It now follows that

$$4\,\mathcal{P}^3 - \mathcal{P}'^2 - 60k_4\,\mathcal{P} = 140\,k_6$$

Elliptic Integrals: The integration of the above equation gives

(46) $$\int_\infty^{\mathcal{P}} \frac{d\,\mathcal{P}}{\sqrt{\{4\,\mathcal{P}^3 - 60\,k_4\,\mathcal{P} - 140k_6\}}} = z$$

We have already seen that $\mathcal{P}$ is a function of z. Hence, this equation gives z as a function of $\mathcal{P}$. In other words this is the inverse function to the elliptic function $\mathcal{P}(z)$. This inverse function is called an elliptic integral. We denote this by

(47) $$z = \mathcal{P}^{-1}(z) = \int_\infty^{w} \frac{d\,w}{\sqrt{\{4\,w^3 - 60\,k_4\,w - 140k_6\}}}$$

Note that such an integral cannot be expressed in closed form in terms of algebraic, trigonometric, inverse trigonometric, logarithmic, and exponential functions unless the cubic has repeated roots. We give below three types of elliptic integral. The integral

(48) $$z = \int_0^{w} \frac{du}{\sqrt{(1-u^2)\,(1-k^2\,u^2)}}, \quad |\,k\,| < 1$$

is called an elliptic integral of the first kind. If w is real and is such that $|\,w\,| < 1$ then the integral exists. Let $u = \sin\theta$ and $w = \sin\varphi$. Then (48) reduces to

$$z = \int_0^{\phi} \frac{d\theta}{\sqrt{\{1 - k^2\,\sin^2\,\theta\}}}$$

If $k = 0$, (48) becomes

$$z = \sin^{-1} w = \int_0^w \frac{du}{\sqrt{\{(1-u)^2\,(1-k^2\,u^2)\}}}$$

The elliptic integral of the second kind is

$$(49) \qquad z = \int_0^w \sqrt{\left\{\frac{1-k^2\,u^2}{1-u^2}\,du\right\}}$$

$$= \int_0^\phi \sqrt{1-k^2\,\sin^2\,\theta\,d\,\theta}$$

The elliptic integral of the third kind is

$$(50) \qquad z = \int_0^w \frac{du}{(1+nu^2)\,\sqrt{\{(1-u^2)\,(1-k^2\,u^2)\}}}$$

$$= \int_0^\phi \frac{d\theta}{(1+n\,\sin^2\,\theta)\,\sqrt{\{1-k^2\,\sin^2\,\theta\}}}$$

68. Show that the Weierstrass zeta function

$$\zeta\,(x) = \frac{1}{z} + \sum_{w \neq 0} \left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2}\right)$$

satisfies

$$\zeta\,(z) = -\int \wp\,(z)\,dz$$

Solution: $\wp\,(z) = -\,\zeta'\,(z) = \dfrac{1}{z^2}\,\sum_{w \neq 0}\left\{\dfrac{1}{(z-w)^2} - \dfrac{1}{w^2}\right\}$

69. Show that the Weierstrass sigma function $\sigma\,(z)$ defined by

$$\sigma\,(z) = z \prod_{w \neq 0}\left(1 - \frac{z}{w}\right) e^{(z/w)+(z^2/2w^2)}$$

satisfies $\qquad \log\,\sigma\,(z) = \int \zeta\,(z)\,dz$

Solution: By logarithmic differentiation we have

$$\frac{\sigma'\,(z)}{\sigma\,(z)} = d_z\,\log\,\sigma\,(z) = \zeta\,(z)$$

70. Show that the constants c_1 and c_2 defined by

$$\zeta\,(z+a_1) - \zeta\,(z) = 2c_1$$
$$\zeta\,(z+a_2) - \zeta\,(z) = 2c_2$$

satisfies the relation

$$a_2\,c_1 - a_1\,c_2 = \pi_i$$

Solution: By integrating $\zeta\,(z)$ over a fundamental parallelogram we get

$$\int \zeta\,(z)\,dz = \int_{a_1+a_2}^{a_2} 2c_2\,\,dz + \int_{a_1}^{a_1+a_2} 2c_1 dz = 2\pi_i$$

71. Show that the entire function $\sigma\,(z)$ satisfies the following relations

$$\sigma\,(z+a_1) = -\,e^{n_1(2z+a_1)}\sigma\,(z)$$

$$\sigma\,(z + a_2) = -\, e^{n_2(2z + a_2)}\sigma\,(z)$$

Solution: We have

$$\zeta\,(z) = \frac{\sigma'\,(z)}{\sigma\,(z)} = d_z \log \sigma\,(z)$$

It now follows from the previous exercise that

$$\log \sigma\,(z + a_1) - \log \sigma\,(z) = 2c_1\, z + k_1$$
$$\log \sigma\,(z + a_2) - \log \sigma\,(z) = 2c_2\, z + k_2$$

i.e.

$$\sigma\,(z + a_1) = \sigma\,(z)\; e^{2c_1 z}\; e^{k_1}$$
$$\sigma\,(z + a_2) = \sigma\,(z)\; e^{2c_2 z}\; e^{k_2}$$

since $\sigma\,(z) = -\,\sigma\,(-\,z)$ is an odd function if $2\,z = -\,a_1$, we find that

$$\sigma\left(\frac{a_1}{2}\right) = \sigma\left(-\frac{a_1}{2}\right) e^{-c_1 a_1} e^{k_1}$$

and

$$e^{k_1} = -\, e^{c_1 a_1}$$

Similar result holds for k_2.

72. Let the zeros and poles of an elliptic function $f\,(z)$ be located at $\{z_k\}$ and $\{p_k\}$ in a fundamental parallelogram. Suppose that

$$\sum_k z_k = \sum_k p_k$$

Then prove that

$$f\,(z) = C \prod_k \frac{\sigma\,(z - z_k)}{\sigma\,(z - p_k)},\ C \text{ is a constant}$$

Solution: Denote the product by $g\,(z)$. It follows from exercise 71 that

$$\sigma\,(z - z_k + a_1) = -\,\sigma\,(z - z_k)\; e^{2c_1\,(z - z_k)}\; e^{c_1\, a_1}$$
$$\sigma\,(z - p_k + a_1) = -\,\sigma\,(z - p_k)\; e^{2c_1\,(z - p_k)}\; e^{c_1\, a_1}$$

Hence

$$g\,(z + a_1) = g\,(z) \exp\left[2C_1 \sum_k\ (p_k - z_k)\right] = g\,(z)$$

Similarly we can obtain

$$g\,(z + a_2) = g\,(z)$$

Thus $g\,(z)$ is an elliptic function. It now follows that $\dfrac{f\,(z)}{g\,(z)}$ is also an elliptic function without any singularity. Hence, it is a constant.

BIBLIOGRAPHY

1. Ahlfors, L.V. *Complex Analysis,* 3rd ed. McGraw-Hill, New York, 1979.
2. Apostol, T.M. *Mathematical Analysis,* 2nd ed. Addison-Wesley, Reading, Mass, 1985.
3. Boas R.P. *Invitation to Complex Analysis,* Random House, New York, 1987.
4. Bartle, R.G. *The Elements of Real Analysis,* 2nd ed. John Wiley & Sons, 1976.
5. Birkhoff, G. and S. Maclane. *A Survey of Modern Algebra,* 4th ed. Macmillan, New York, 1977.
6. Cartan, H. *Elementary Theory of Analytic Functions of One or Several Complex Variables,* Addison and Wesley, New York, 1963.
7. Conway, J.B. *Functions of One Complex Variable,* Springer Verlag, New York, 1973.
8. Davis, P.J. *Interpolation and Approximation,* Dover Publications, New York, 1975.
9. Henrici, P. *Applied and Computational Complex Analysis,* Vol. 3. Wiley, New York, 1986.
10. Hille, E. *Analytic Function Theory*, Ginn & Co. Boston, 1959 (Vol. 1), 1962 (Vol. 2). Reprint. Chelsea. New York, 1974.
11. Lang, S. *Complex Analysis*, Addison-Wesley, 1977.
12. Markushevich, A.I. Theory of Functions of a Complex Variable. Vol. 1, Translated by R.A. Silverman, Prentice Hall, 1965.
13. Rudin, W. *Real and Complex Analysis,* 2nd ed. Tata McGraw-Hill, New Delhi, 1981.
14. Rudin, W. *Principles of Mathematical Analysis,* 3rd ed. McGraw-Hill, New York, 1976.
15. Watson, G.N. *A Treatise on the Theory of Bessel Functions,* Cambridge Press, 1945.

LIST OF SYMBOLS

$\in$	belongs to
$\notin$	does not belong to
$\subset$	is a subset of, is contained in
$\supset$	contains
$\cup$	union
$\cap$	intersection
$\Rightarrow$	implies
$\Leftrightarrow$	logical equivalence, if and only if
$A = B, A \neq B$	equality and inequality of sets
$A - B$	the set of all points in A which are not in B
$A \Delta B$	symmetric difference of two sets
A^c	the complement of A
$A \times B$	Cartesian product of A and B
$[a, b], (a, b]$, *etc*	intervals on the real line
$B_r(a)$	Open ball with radius r and centre a
$B(X)$	Space of all bounded real valued functions on X
C	Complex number system, complex plane
C_∞	extended complex plane
$\mathbb{C}(G, \Omega)$	set of all continuous functions from G to Ω
$d(x, y)$	distance from one point to another
$d(x, A)$	distance from a point to a set
diam S	diameter of the set S
$D(z_1, r)$	Open circular disc with radius r and centre z_1
$\overline{D}(z_1, r)$	the closure of $D(z_1, r)$
$D'(z_1, r)$	punctured disc with radius r and centre z_1
$D_r(a)$	open circular disc with radius r and centre a
δ_{ij}	Kronecker delta
∂T	the boundary of the rectangle T
ϕ	empty set

$f : X \to Y$	function (or mapping) with domain X and range in Y
$f^{-1}: Y \to X$	inverse function (or mapping)
$f(E)$	image of a set under a mapping
f/E	the restriction of a function on E
$f = g$	equality for mappings
$gof : X \to Y$	composition of mappings
I_X	identity mapping on a set
int. A	interior of a set
$\cap_i A_i$, etc.	intersection of a class of sets
$-\infty, +\infty$	infinity (minus and plus)
iff	if and only if
inf A	infimum (or greatest lower bound) of a set of real numbers
lim x_n	limit of a sequence
lim sup or lim	limit superior or upper limit
lim inf or lim	limit inferior or lower limit
lim $f(x)$	limit of the function
R	real number system
$R \times R$ (or R^2)	co-ordinate plane
R^n	n-dimensional Euclidean space
$Re\, z$	real part of the complex number z
$\overline{S}$	the closure of a set S
S°	the interior of a set S
S'	the derived set
∂S	the boundary of a set
sup A	supremum (or least upper bound) of a set of real numbers
$\cup_i A_i$, etc.	union of a class of sets
$\lVert \; \rVert$	norm
$A \approx B$	the set A is equivalent to the set B
$\langle f, g \rangle$	inner product of f with g
w_f	modulus of continutiy of the function
$\lVert x \rVert$	norm of x
$\lvert z \rvert$	magnitude (or absolute value) of the complex number z
$\overline{z}$	conjugate of the complex number z
$z_2 \equiv z_1$ (mod. 2 W)	congruence modulo 2 W for complex numbers
$\Gamma(z)$	the Gamma function
$\wp(z)$	the Weierstrass function
$\zeta(z)$	the Weierstrass Zeta function
$\sigma(z)$	the Weierstrass sigma function
$\prod_{n=1}^{\infty} z_n$	infinite product of complex numbers

INDEX

A

- Abelian 187
- Absolute value 90
- Accumulation point 22
- Addition theorem 194
- Algebra 194
- Analytic automorphism 101
  - continuation 202
  - function 202
  - isomorphism 101
  - square root 241
- Annulus 94
- Approximation 225
- Associative law 15
- Automorphism 101

B

- Bernstein polynomial 226
  - theorem 226
- Bessel's differential equation 220
  - function 220
- Beta function 184, 280
- Blaschke product 171
- Boundary 179
  - point 246
- Bounded entire function 74
  - set 140
  - variation 51

C

- Canonical product 165
- Cantor set 25
- Capacitor 153
- Cardinal number 5
- Cartesian product 3, 294
- Cauchy
  - Riemann equations 40
  - sequence 24, 33, 34, 78, 80, 82, 158, 159
- Cauchy's inequality 74, 229
  - integral formula 71, 72, 94, 95, 131
  - theorem 53, 59, 63, 75, 270
- Chain 39, 69, 70, 94, 114, 204, 242, 244
- Circle 136
  - of convergence 84
- Closed ball 22
  - chain 70, 264
  - mapping 28
  - path 55
  - polygon 64
  - set 233
  - square 231
- Closure 294
- Compact 249
  - metric space 27
  - set 34
- Complement 233
- Complete metric space 24
- Complex number 4
- Complex plane 37
- Composition of transformation 123
- Conductor 153
- Conformal 238
  - equivalence 241
  - mapping 270

Congruent 188
Conjugate 295
  harmonic function 272
Connected set 29
Continuous function 31
  second partial derivative 41
Continuously differentiate 41
Contour 77
Convergence 81
Convergent product 161
  sequence 24
Convex region 32
  set 32
Convexity theorem 180, 181
Convolution 143
Coordinate plane 10
Countable 16
Cross-ratio 126
Complex valued 43
  composition 5
  continuous 38

D

Dense set 23
  everywhere 46
  nowhere 209
Derived set 22
Diagonal sequence 240
Dilation 122
Diameter 205
Differentiable 250
  path 50
Differential equation 220
Dirac family 143
  sequence 143
Dirichlet problem 146
Discrete 19
Disjoint region 231
Disk 234, 237
Distributive laws 9
Doubly periodic 188

E

Electrostatics 147
Element 1
Elementary factor 163
Elliptic function 187
Entire complex plane 269
  function 291
Equipotential 148
Essential singularity 97
Euler's formula 185
Extended complex plane 126

F

Field 153
Finite order 173
Finite rectangle 246
Fixed point 271
Fluid density 151
  flow 151
Fourier series 142
Fundamental parallelogram 187
  theorem of algebra 74
  theorem of calculus 53

G

Gamma function 171
Gaps 210
Generating function 220
Geometric series 287
Global approximation 230
Group 133

H

Hadmard
  gap theorem 228, 230
  product 187
  product representation 187
  three circle theorem 195
Half plane 106
Hankel's integral 172
Harmonic 176
Harnack inequality 155
Heine-Borel theorem 209
Homeomorphic 29
  image 33
Homeomorphism 101
Homogeneous 152
Homologous 243

Homotopic 255
  to a point 59
Hyperbola 119
Hyperbolic function 44, 45

I

Idempotent 2
Identity transformation 124
Image 128
Index 144
Infinite open polygon 130
  product 161
  ray 121
  rectangle 246
  series 79
Infinitely differentiable 138
Initial point 203
Insulated boundary 152
Interior 203
  point 231
Intersection of sets 2
Invariant 101
Inverse image 266
  transformation 129
Inversion 122
Irregular singularity 213
Isolated 237
  essential singularity 97
  singularity 97
Isomorphism 101
Isothermal 152

J

Jacobi polynomial 219
Jacobian 135
Jordan curve 231

L

L_1-norm 78
L_2-norm 78
Laplace's equation 138
Lattice 187
Laurent expansion 284
  series 90
Legendre's differential equation 221
Lim inf 81
  sup 81
Limit 81
  point 97
Line integral 52, 53
  segment 62
Linear fractional transformation 122
Liouville's theorem 74, 75, 241, 256
Local primitive 55
Locally analytic function 230
Logarithm 290
Logarithmic function 45
  residue 113

M

Mapping 3
Markushevich 235
Maximum modulus theorem 238
Mean value theorem 41
Mellin transform 183
Meromorphic 188
Metric space 19
Minimum principle 140
Mittag-Leffler 166
Möbius transformation 238
Modulus of continuity 53
Morera's theorem 75
Multiplicity 96

N

Natural boundary 202
Near-together 244
Neighbourhood 272
Neumann's function 221
Non-viscous 147
Norm 34
Normal family 239, 242
Normed space 34
Nowhere analytic 252

O

One-to-one correspondence 5
Open ball 19
Ordinary point 213

P

Parabola 135
Parallelogram law 13
Partial sum 79
Partition 203
Periodic 287
Perpendicular bisector 87
Phragmen-Lindelöf theorem 179
Picard 98
Piecewise differentiable 50
Poisson formula 141
Pole 166
Polygon 30
Polygonal curve 30
Polynomial 64
Power series 79
Prime number 287
Primitive 53
Principal part 95
  value 11
Principle of symmetry 128
Proper subset 1
Punctured disk 94

Q

Quadrant 119

R

Radius of convergence 162
Ratio test 81
Rational function 98
Rectangle 104
Rectangular chain 245
  closed chain 245
  curve 243
  domain 259
  path 244
  region 246
Rectifiable path 52
Recurrence relation 222
Recursion formula 220
Reflexive 5
Region 32
  of convergence 89
Regular singularity 213
Removable singularity 96
Residue 102
  theorem 102
Restriction 159
Riemann
  mapping theorem 237
  sphere 14
  zeta function 89
Rodrigues' formula 221
Root-test 90
Rouche's theorem 114
Runge 230
Runge's theorem 230

S

Schwarz-Christoffel transformation 272
Schwarz reflection principle 204
Sector of angle 129
Segment 180
Semi-circle 164
Sequence 6
  of functions 82
Sequentially compact 26
Series of functions 82
Sets 1
Simply connected 62
Singular point 107, 169
Singularity 154
Sink 150
Source 150
Stagnation point 148
Stereographic projection 14
Stirling's formula 185
Stream function 148
Subharmonic function 155
Subinterval 225
Subsequence 239
Subset 1
Sup norm 35
Supremum 53
Symmetric 126
Symmetry 126

T

Taylor part 95
  series 95
Temperature 152
Totally bounded set 27
Transcendental function 42
Translation 122
Transitive 5
Triangle inequality 12
Trigonometric function 45

U

Unbounded region 148
Uniform approximation 225
  continuity 26
  convergence 82
Uniformly bounded 239
Union of sets 202
Unit circle 14
  element 8
Upper half plane 106

V

Vector space 34
Velocity potential 147
Vivanti-Pringsheim theorem 207

W

Walsh 229
Weight factor 235
Weierstrass approximation theorem 229
  elliptic function 287
  factorization theorem 159
  function 191
  M test 83
  sigma function 198
  zeta function 291
Winding number 66

Z

Zero element 8
  of order *n* 92